Part II

第二部分

Perturbative Quantum Gravity

微扰量子引力

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The Background Information About Perturbative Quantum Gravity

微扰量子引力背景知识

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Abstract

摘要

The purpose of this chapter is to give a general introduction and status review on the perturbative approach to quantum gravity (QG). This text is a modified version of the corresponding chapters of Part II of the recent textbook on quantum field theory (QFT) and QG, co-authored with I.L. Buchbinder and published in Oxford University Press. We discuss the choice of the starting action in the QG models; degrees of freedom and propagator of metric perturbations; power counting and renormalizability of these models; the problems related to higher-derivative theories and ghosts, such as quantum unitarity and the stability of classical solutions in general relativity; and the perspective to overcome these problems. The gauge-fixing and parametrization dependencies are discussed in detail using the corresponding general QFT theorems developed in gauge theories. On top of that, we present a basic example of deriving the one-loop divergences and discuss an important example of the renormalization group in QG. The gauge invariant renormalizability of QG is considered in another chapter of the handbook, written together with P.M. Lavrov.

本章旨在对量子引力 (QG) 的微扰方法做综述性介绍与现状梳理。本文是近期量子场论 (QFT) 与量子引力教材第二部分对应章节的修改版本，该教材由我与 I.L. Buchbinder 合著，牛津大学出版社出版。我们讨论量子引力模型中初始作用量的选择、度规扰动的自由度与传播子、这类模型的幂次计数与可重整性；高导数理论和鬼场相关的问题，例如广义相对论中的量子幺正性与经典解稳定性；以及解决这些问题的研究前景。我们借助规范场论中已发展的对应一般性 QFT 定理，对规范固定与参数化依赖问题展开了详细讨论。在此基础上，我们给出推导单圈发散的基础示例，并讨论量子引力中重整化群的一个重要实例。量子引力的规范不变可重整性在本手册由我与 P.M. Lavrov 合著的另一章中讨论。

Keywords

关键词

Quantum gravity · Gravitational action · Gauge fixing · Propagator · Power counting - Renormalizability - Higher derivatives - Ghosts - Stability

量子引力 · 引力作用量 · 规范固定 · 传播子 · 幂次计数 - 可重整性 - 高阶导数 - 鬼场 - 稳定性

Introduction

引言

Perturbative quantum gravity (QG) aims to construct quantum gravitational theory in a manner close to how QFT describes other fundamental forces, such as electroweak and high-energy strong interactions. One of the main purposes of perturbative QG is to get the quantum corrections to the classical (tree-level) action of gravity and the corresponding effective equations of motion for the metric.

微扰量子引力 (QG) 旨在以近似量子场论 (QFT) 描述电弱相互作用、高能强相互作用等其他基本力的方式构建量子引力理论。微扰量子引力的核心目标之一是得到引力经典 (树图阶) 作用量的量子修正，以及对应度规的有效运动方程。

The importance of loop corrections to the gravitational equations, coming from QG or from the quantum effects of matter fields (semiclassical approach), is partially owing to the fact that, at some point, we hope to be able to compare these corrections with experimental or observational data. Another reason to study QG is that the consistency of quantum theory may be useful to establish the restrictions on the modifications of general relativity (GR). It is assumed we may be able to select those gravity theories that are consistent at the quantum level and discard other models. Independent of all these reasons, in the last six decades the perturbative QG became an important part of the general QFT scenario and, especially, of the gauge field theories.

引力圈修正 (来源于量子引力，或物质场的量子效应，即半经典方法) 的重要性部分源于: 我们期望最终能将这些修正与实验或观测数据进行比对。研究量子引力的另一原因在于，量子理论的自洽性或许有助于对广义相对论 (GR) 的修正方案给出约束。我们有望筛选出量子层面自洽的引力理论，排除其他模型。抛开这些原因不谈，近六十年来，微扰量子引力已成为整个量子场论框架，尤其是规范场论中十分重要的组成部分。

When initiating the QG, we meet two main choices representing kinds of the points of bifurcation, where one can choose the direction of how to construct the theory. The first question is to decide what should be the object of quantization. In the traditional perturbative quantum gravity, we choose to quantize the spacetime metric and apply to it the rules of quantization that are common to all gauge theories, such as the Yang-Mills. It is worth mentioning that one can choose other quantum variables to describe gravity, e.g., the tetrad, or consider the connection independent of the metric (first-order formalism). Owing to the size and content limitations, we will not discuss these possibilities in what follows.

开展量子引力研究时，我们会遇到两个核心选择，它们是构建理论路径的分叉点。第一个问题是确定量子化的对象。在传统微扰量子引力中，我们选择对时空度规进行量子化，并对其套用适用于所有规范理论 (如杨-米尔斯理论) 的通用量子化规则。值得一提的是，也可以选择其他量子变量来描述引力，例如标架，或是考虑独立于度规的联络 (一阶形式)。受篇幅和内容限制，下文不对这些可能性展开讨论。

On the other hand, it is possible to extend the set of the quantum fields, including other fields along with the metric, or even replacing the metric. One of the most interesting approaches to make such extension is to provide an extended symmetry, as it is done in supergravity. We can refer the reader to the corresponding section of the present handbook for the reviews of supergravity, but will not discuss it here. Another possibility is to add quantum matter fields. Let us note that there is a reduced, semiclassical approach, when only the matter fields are assumed quantum and metric is regarded as a classical background. This kind of theories shares many technical and conceptual features with perturbative QG, so the reader may find useful to study the semiclassical approach first. There are many useful monographs on this subject, let us mention just a few of them, [1-6] and also the recent textbook [7], where the reader can also find an introduction to QG (For the interested reader, there are good books and reviews of semiclassical and quantum gravity theories, e.g., [8-11]. One can find many other useful sources starting from these references.). In what follows, we shall address the same subjects which are discussed in Sects. 18-21 of the textbook [7]. Most of the present chapter can be seen as a modified and more review-style extract from this book.

另一方面，我们可以扩展量子场集，在度规之外纳入其他场，甚至用其他场替换度规实现量子化。实现这类扩展最有意思的思路之一是引入扩展对称性，超引力就是这类方案的典型代表。本手册已有对应章节对超引力做综述，因此本文不展开讨论。另一种可能是引入量子物质场。请注意，还存在一个简化的半经典方案: 该方案仅假设物质场是量子的，度规仍视为经典背景。这类理论与微扰量子引力共享许多技术和概念特征，因此读者先学习半经典方法会有所助益。关于这一主题已有诸多有用专著，此处仅列举 [1-6]，以及近期的教科书 [7]，其中读者也可以找到量子引力的导论 (感兴趣的读者可以参考介绍半经典引力和量子引力理论的优秀著作与综述，例如 [8-11]，从这些参考文献出发还能找到更多其他有用资料)。下文我们将讨论教科书 [7] 第 18-21 节涉及的相同主题，本章大部分内容都可以视为该书内容经改编后形成的综述性摘录。

The second important choice appears after we agree to quantize the metric. One has to choose the classical theory which should serve as a basis for the perturbative quantum gravity. Obviously, such initial model may be Einstein's GR; however, there are strong reasons to try also other models of gravity. The choice of a model defines many important features of the theory, including its particle contents, renormalizability, unitarity, stability of classical solutions, and high-energy (UV) and low-energy (IR) behaviors, as we shall discuss below.

第二个重要选择出现在我们确定对度规进行量子化之后: 需要选择一个经典理论作为微扰量子引力的基础。显然，这个初始模型可以是爱因斯坦广义相对论，但也有充分理由尝试其他引力模型。正如下文我们会讨论到的，模型的选择决定了理论的诸多核心性质，包括粒子谱、可重整性、幺正性、经典解稳定性，以及高能 (UV) 和低能 (IR) 行为。

In what follows, we review the main elements of the perturbative QG, referring to the recent textbook [7] for most of the technical details. First of all, we consider the choice of the action and the corresponding gauge-fixing conditions, mostly, in the framework of the background field method [1, 12] (the reader can find more references starting from these two). After that, we review the structure of propagator of the metric perturbations in various models and the corresponding analysis of the degrees of freedom in QG.

下文我们将综述微扰量子引力的核心要素，大部分技术细节参见近期教科书 [7]。首先，我们主要在背景场方法的框架下讨论作用量的选择和对应的规范固定条件 [1, 12] (读者可以从这两份资料出发找到更多参考文献)。随后，我们综述不同模型中度规扰动传播子的结构，以及对量子引力自由度的对应分析。

The proof of the gauge-invariant renormalizability in QG is left for > Chap. 9, "Gauge Invariant Renormalizability of Quantum Gravity". This analysis possesses a higher level of complexity compared to the mentioned textbook, but, as a result, the main statements are obtained in a more general and more concrete way. The main two outputs of this consideration, based on the BRST symmetry and the Batalin-Vilkovisky technique, are the following two statements:

量子引力中规范不变可重整性的证明留待 > 第 9 章《量子引力的规范不变可重整性》讨论。该分析的复杂度高于上述教科书，但作为结果，核心结论的得出更具一般性，也更明确。基于 BRST 对称性和巴塔林-维尔可夫斯基技术，该研究得到的两个核心结论如下:

(i) The counterterms in the generally covariant theory of QG may have the same symmetry, i.e., the diffeomorphism invariance, as the initial classical action. This symmetry holds at the quantum level if we use a regularization preserving this symmetry, such that the general covariance is not violated by quantum corrections. For instance, this is the case for the perturbative QG in even dimension $n = 2m + 2$ (where $m = 1, 2, \ldots$), including $n = 4$; the last will be used by default in the rest of this chapter.

(i) 广义协变量子引力理论中的抵消项可与初始经典作用量具有相同对称性，即微分同胚不变性。若我们采用能保留该对称性的正规化方案，量子修正就不会破坏广义协变性，该对称性在量子层面依然成立。例如，偶数维微扰量子引力就属于这种情况 $n = 2m + 2$ (其中 $m = 1, 2, \ldots$)，包括 $n = 4$ ；本章余下部分默认采用该设定。

(ii) The dependence on the choice of the gauge fixing and on the parametrization of quantum field, at any order of loop expansion, is proportional to the effective equations of motion. As an important consequence of this rule, the respective dependencies in the one-loop divergences vanish on the classical mass shell.

(ii) 在圈展开的任意阶，规范固定选择和量子场参数化的依赖关系都与有效运动方程成正比。该规则的一个重要结论是: 单圈发散中的对应依赖关系在经典质壳上为零。

The first feature implies that the general structure of divergences in any loop order may be defined on the basis of power counting of the Feynman diagrams, which means it can be reduced to the use of the dimensional arguments. In this way we can, in most of the cases, say that the given model of QG is renormalizable, non-renormalizable, or super-renormalizable and even establish the structure of possible divergences in any loop order - without making explicit calculations. The remaining questions concern only the coefficients of the given terms in the divergences.

第一个特征表明，任意圈阶发散的一般结构可基于费曼图的幂次数计数确定，也就是说可以简化为量纲分析。通过这种方式，大多数情况下我们无需显式计算，就能判断给定量子引力模型是可重整、不可重整还是超可重整，甚至能确定任意圈阶可能存在的发散结构。余下问题仅涉及发散中给定项的系数。

The second feature is also important for practical purposes, for the following two reasons. From one side, one can choose the gauge-fixing condition in such a way that makes the calculations technically simpler. And, on another hand, we can extract that part of the quantum corrections which are gauge-fixing and parametrization invariant and, in principle, consider this as a physical output of the loop calculations.

第二个特征出于以下两个原因，对实际应用也十分重要。一方面，我们可以选择技术上简化计算的规范固定条件；另一方面，我们可以提取出量子修正中不依赖规范固定和参数化的不变部分，原则上可将其作为圈计算的物理输出。

The chapter is organized as follows. In section ”Models of QG: General Classification and Gauge Fixing” we formulate the most relevant models of QG that may constitute a sound basis of the perturbative treatment. Let us note that we only slightly touch the nonlocal models which represent, nowadays, a popular object of studies. The reason is that there is a special section devoted to the nonlocal theories and we do not like to have repetitions. Thus, we mainly consider the QG based on GR and on the different kinds of local polynomial models. In section ”Bilinear Forms and Linear Approximation,” one can find the bilinear expansions of all metric-dependent terms that define the propagator of the quantum metric. In section ”Propagator of Metric and the Barnes-Rivers Projectors” we describe these propagators different models of QG. Next, section ”Gauge-Invariant Renormalization in Quantum Gravity” is briefly formulating the the main statements about gauge-invariant renormalization in QG, while the detailed discussion is postponed to the separate chapter of this section. Section ”Power Counting and Classification of Quantum Gravity Models” describes the power counting and classification of the QG models to renormalizable, non-renormalizable, and super-renormalizable ones. Section ”Massive Ghosts in Higher-Derivative Models” discusses the problem of ghosts in higher-derivative QG. This problem certainly represents the main difficulty of all the QG program. We give a basic introduction and a brief report on the existing results in this area. In section ”Gauge-Fixing Dependence Using General Formalism” we describe the gauge-fixing and parametrization dependencies of the one-loop effective action and section ”One-Loop Divergences in Quantum GR” shows the detailed derivation of the one-loop divergences in quantum GR, in the simplest parametrization and gauge fixing. Section ”On-Shell Renormalization Group in Quantum GR” discusses an interesting example of the renormalization group applied to the quantum GR in a manner that fits the effective approach to QG. There is a special section of the present handbook, devoted to the subject of effective QG, so we do not go too deep into the subject and only show this particular example. Section "One-Loop Divergences in Other Models of QG" gives a short review of the one-loop calculations in other models of QG. Owing to the size limitations, this part is very much incomplete and does not cover the extensive literature on the subject, so it is included just as a starting point for the interested readership. Finally, in section "Concluding Discussion" we draw our conclusions and present final discussions of the current situation in perturbative QG.

本章结构安排如下: 在“量子引力模型: 一般分类与规范固定”一节中，我们阐述了可作为微扰处理可靠基础的最相关量子引力模型。需要说明的是，我们仅略微涉及非定域模型——这类模型是当下的热门研究对象，原因在于本手册已有专门章节讨论非定域理论，我们不打算重复内容。因此我们主要研究基于广义相对论的量子引力以及各类局部多项式模型。在“双线性形式与线性近似”一节中，读者可以找到所有依赖度规项的双线性展开，这些项定义了量子度规的传播子。在“度规传播子与巴恩斯-里弗斯投影算子”一节中，我们描述了不同量子引力模型中的这些传播子。随后，“量子引力中的规范不变重整化”一节简要给出了量子引力规范不变重整化的核心结论，详细讨论留到本部分的单独章节展开。“幂次数计数与量子引力模型分类”一节描述了幂次数计数，并将量子引力模型分为可重整、不可重整与超可重整三类。“高阶导数模型中的有质量鬼场”一节讨论了高阶导数量子引力中的鬼场问题，该问题无疑是整个量子引力研究项目的核心难点。我们对该领域的现有成果给出基础介绍与简要总结。在“利用一般形式化分析规范固定依赖性”一节中，我们描述了单圈有效作用量对规范固定和参数化的依赖关系；“量子广义相对论的单圈发散”一节给出了最简参数化与规范固定下，量子广义相对论单圈发散的详细推导。“量子广义相对论的壳上重整化群”一节讨论了一个有趣的例子: 重整化群应用于量子广义相对论，其方式契合量子引力的有效场论方案。本手册已有专门章节讨论有效量子引力这一主题，因此我们不会深入探讨，仅给出这个具体示例。“其他量子引力模型中的单圈发散”一节简要回顾了其他量子引力模型中的单圈计算。受篇幅限制，这部分内容十分不全，没有涵盖该主题的大量现有文献，仅作为供感兴趣读者入门的参考。最后，在“结论与讨论”一节中，我们给出总结，并最终讨论微扰量子引力的研究现状。

In the rest of the chapter, we use DeWitt notations, when the covariant integral $\int d^4x\sqrt{-g} = \int_x$ may be assumed but not written explicitly when this is obvious. We denote functional trace Tr and determinant Det, which includes the same covariant integration over spacetime coordinates, and use pseudo-Euclidean notations such as $\sqrt{-g}$, even in case of using the heat-kernel technique, which requires Euclidean metric. In these cases, we assume the Wick rotation, regardless it may be a nontrivial issue in some models of QG. On top of this, the notations include the signature $\eta_{\alpha\beta} = \mathrm{diag}\,(+---)$, the definition of the Riemann tensor

本章余下部分我们采用德维特记号，协变积分 $\int d^4x\sqrt{-g} = \int_x$ 在明显情况下可不写出。我们用符号 Tr 表示泛函迹，Det 表示泛函行列式，二者均包含对时空坐标的相同协变积分；同时采用类伪欧记号，如 $\sqrt{-g}$ ，即便在要求欧氏度规的热核技术中也如此。在这类情况下我们默认已做威克转动，不特意讨论它在某些量子引力模型中可能是个非平凡问题。此外，我们的记号还包括号差 $\eta_{\alpha\beta} = \mathrm{diag}\,(+---)$ ，以及黎曼张量的定义

$$R^{\lambda}_{\tau\alpha\beta} = \partial_\alpha \Gamma^{\lambda}_{\tau\beta} - \partial_\beta \Gamma^{\lambda}_{\tau\alpha} + \Gamma^{\lambda}_{\gamma\alpha}\Gamma^{\gamma}_{\tau\beta} - \Gamma^{\lambda}_{\gamma\beta}\Gamma^{\gamma}_{\tau\alpha}, \tag{1}$$

Ricci tensor $R^{\alpha}_{\mu\alpha\upsilon} = R_{\mu\upsilon}$, and its trace $R = R_{\mu\upsilon}g^{\mu\upsilon}$, that is, Ricci scalar.

里奇张量 $R^{\alpha}_{\mu\alpha\upsilon} = R_{\mu\upsilon}$ ，其迹 $R = R_{\mu\upsilon}g^{\mu\upsilon}$ 即里奇标量。

Models of QG: General Classification and Gauge Fixing

量子引力模型: 一般分类与规范固定

Our purpose is to construct the quantum theory of the metric field $g_{\mu\upsilon}$. In GR, the metric is subject to the gauge transformation, also called diffeomorphism, corresponding to the infinitesimal coordinate transformation

我们的目标是构建度量场 $g_{\mu\upsilon}$ 的量子理论。在广义相对论中，度量满足对应于无穷小坐标变换的规范变换，也称为微分同胚

$$x^{\mu} \to x'^{\mu} = x^{\mu} + \xi^{\mu},\ \xi^{\mu} = \xi^{\mu}(x). \tag{2}$$

Let us start by deriving the diffeomorphism transformation for the metric. Keeping only the terms of the first order in ξ^{μ} and their derivatives, we get

我们先从推导度量的微分同胚变换开始。仅保留 ξ^{μ} 及其导数的一阶项，我们得到

$$g'_{\alpha\beta}(x) = g'_{\alpha\beta}(x') - \frac{\partial g'_{\alpha\beta}(x')}{\partial x'^{\lambda}}\xi^{\lambda} = g'_{\alpha\beta}(x') - \partial_{\lambda}g_{\alpha\beta}\xi^{\lambda}.$$

Using the tensor transformation rule,

利用张量变换规则,

$$g'_{\alpha\beta}(x') = \frac{\partial x^{\rho}}{\partial x'^{\alpha}}\frac{\partial x^{\sigma}}{\partial x'^{\beta}}g_{\rho\sigma}(x) = \left(\delta^{\rho}_{\alpha} - \partial_{\alpha}\xi^{\rho}\right)\left(\delta^{\sigma}_{\beta} - \partial_{\beta}\xi^{\sigma}\right)g_{\rho\sigma}(x)$$

$$= g_{\alpha\beta} - g_{\rho\beta}\partial_{\alpha}\xi^{\rho} - g_{\alpha\rho}\partial_{\beta}\xi^{\rho} \tag{3}$$

and taking the two expressions together, we get

将两个表达式结合，我们得到

$$\delta g_{\alpha\beta}(x) = g'_{\alpha\beta}(x) - g_{\alpha\beta}(x) = -g_{\lambda\beta}(x)\partial_{\alpha}\xi^{\lambda}(x) - g_{\alpha\lambda}(x)\partial_{\beta}\xi^{\lambda}(x)$$

$$-\partial_{\lambda}g_{\alpha\beta}(x)\xi^{\lambda}(x) = -\nabla_{\alpha}\xi_{\beta} - \nabla_{\beta}\xi_{\alpha} \tag{4}$$

that defines the generators of the gauge transformations

它定义了规范变换的生成元

$$\delta g_{\mu\upsilon}(x) = R_{\mu\upsilon\lambda}(g)\xi^{\lambda},\ R_{\mu\upsilon\lambda}(g) = -g_{\mu\lambda}\nabla_{\upsilon} - g_{\upsilon\lambda}\nabla_{\mu} \tag{5}$$

We assume that any sort of the QG candidate action $S = S(g)$ possesses the symmetry under (5), i.e., satisfies the Noether identity,

我们假设任意类型的量子引力候选作用量 $S=S(g)$ 都满足 (5) 式下的对称性，即满足诺特恒等式，

$$\frac{\delta S}{\delta g_{\mu\nu}}R_{\mu\nu\lambda}(g)=0. \tag{6}$$

In Chap. 9, "Gauge Invariant Renormalizability of Quantum Gravity", (see also [7]), it is shown that QG is the gauge theory of the Yang-Mills type, which means the algebra of the generators is closed off shell. This feature represents the basis of the two statements (i) and (ii), mentioned in the Introduction.

在第 9 章“量子引力的规范不变可重整性”中 (也可见文献 [7])，已经证明量子引力是杨-米尔斯型的规范理论，这意味着生成元代数脱壳封闭。该性质是引言中提到的两个结论 (i) 和 (ii) 的基础。

In many situations, it is useful to parameterize the metric as a perturbation over the Minkowski spacetime

在很多情形下，将度量参数化为闵可夫斯基时空上的微扰是很有用的

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu} \tag{7}$$

and use the following representation of $h_{\mu\nu}$ (see [11] and section "Propagator of Metric and the Barnes-Rivers Projectors" for more details):

并使用 $h_{\mu\nu}$ 的如下表示 (更多细节见文献 [11] 和“度量传播子与巴恩斯-里维斯投影算子”一节):

$$h_{\mu\nu}=\bar{h}^{\perp\perp}_{\mu\nu}+\partial_\mu\varepsilon^\perp_\nu+\partial_\nu\varepsilon^\perp_\mu+\partial_\mu\partial_\nu\varepsilon+\frac{1}{4}h\eta_{\mu\nu}. \tag{8}$$

In this expression, the tensor component (spin-2 mode) is traceless and transverse, i.e., $\bar{h}^{\perp\perp}_{\mu\nu}\eta^{\mu\nu}=0$ and $\partial^\mu\bar{h}^{\perp\perp}_{\mu\nu}=0$. The irreducible vector component (spin-1 mode) satisfies the condition $\partial_\mu\varepsilon^{\perp\mu}=0$. There are also two scalar fields (or modes) ε and h . The indices are raised and lowered with the flat metric.

该表达式中，张量分量 (自旋 2 模式) 是无迹且横向的，即 $\bar{h}^{\perp\perp}_{\mu\nu}\eta^{\mu\nu}=0$ 和 $\partial^\mu\bar{h}^{\perp\perp}_{\mu\nu}=0$ 。不可约矢量分量 (自旋 1 模式) 满足条件 $\partial_\mu\varepsilon^{\perp\mu}=0$ 。此外还有两个标量场 (或模式) ε 和 h 。指标用平直度量升、降。

In the rest of this section, we describe the most popular actions which are used for constructing the QG models. This description includes the proper action and the details of the DeWitt-Fadeev-Popov (or Fadeev-Popov) procedure required for the Lagrangian quantization of the model. For the sake of generality, we perform the consideration of these procedures using the background field method. The last means that, instead of (7), the expansion is performed around an arbitrary background metric,

在本节剩余部分，我们介绍构建量子引力模型最常用的几种作用量。介绍内容包括相应作用量，以及该模型拉格朗日量子化所需的德维特-法捷耶夫-波波夫 (或称法捷耶夫-波波夫) 步骤的细节。为了一般性，我们采用背景场方法讨论这些步骤。这意味着，展开是围绕任意背景度量进行的，而非 (7) 式的形式，

$$g_{\mu\nu}\to g'_{\mu\nu}=g_{\mu\nu}+h_{\mu\nu}. \tag{9}$$

The interested reader can find more details of the expansion (9) and useful exercises in the book [7].

感兴趣的读者可以在文献 [7] 中找到展开式 (9) 的更多细节以及有用的练习。

DeWitt-Faddeev-Popov Method for QG

量子引力的德维特-法捷耶夫-波波夫方法

Let us briefly sketch the Faddeev-Popov (or DeWitt-Faddeev-Popov) method for QG. There is no critical difference with other gauge models, and we shall use the notations close to the general ones. Namely, we shall denote $g^i = g_{\mu\nu}$ and keep in mind that, depending on our intentions, g^i may be also used for $h_{\mu\nu}$, as defined in (9). Then the transformation rule and the generator from Eq. (5) will be denoted as

我们来简要概述量子引力的法捷耶夫-波波夫 (即德维特-法捷耶夫-波波夫) 方法。它与其他规范模型没有本质区别，我们将采用与通用 notation 相近的记号。具体来说，我们记为 $g^i = g_{\mu\nu}$ ，需要注意的是，根据我们的研究目的，g^i 也可用于表示 (9) 式定义的 $h_{\mu\nu}$ 。此时 (5) 式中的变换规则和生成元记为

$$\delta g^i = R^i_\alpha \xi^\alpha \quad \text{and} \quad R^i_\alpha = R_{\mu\nu\alpha}. \tag{10}$$

The last note about notations is that g^i may be also used for other parameterizations of the quantum metric (An example is Eq. (132) below), and the formulas can be modified accordingly.

关于记号的最后一点说明: g^i 也可用于表示量子度规的其他参数化形式 (例见下文 (132) 式)，公式可相应修改。

The starting point is the naive expression for the functional integral over the quantum metric

我们的出发点是量子度规泛函积分的朴素表达式

$$Z = \int dg e^{iS(g)} \tag{11}$$

where we use $g \equiv g^i$ for the arguments of a functional S and the integration variable, to make formulas more readable. We can generalize the last integral to the generating functional of Green functions by replacing $S \to S + gJ$ and assuming that the source term $gJ = \int_x g^i J_i$ is diffeomorphism invariant. As this replacement does not cause real changes, we shall work with the formula (11).

其中我们用 $g \equiv g^i$ 表示泛函 S 的宗量和积分变量，以让公式更易读。我们可以通过替换 $S \to S + gJ$ ，并假设源项 $gJ = \int_x g^i J_i$ 是微分同胚不变的，将上述积分推广为格林函数的生成泛函。由于这一替换不会带来实质变化，我们将基于 (11) 式展开推导。

The space of integration g^i includes the orbits, i.e., the subspaces defined by the gauge transformations of the metric field (10). Since the action remains constant over any such subspace, it is cleat that each of these subspaces contributes infinite to the integral, which is, therefore, badly defined. Such a divergence is similar to taking the logarithm of determinant of a degenerate matrix. This divergence is a direct consequence of the gauge invariance and represents the problem solved by the DeWitt-Faddeev-Popov method [13, 14] in gravity and in the Yang-Mills theory (see also, e.g., [15] for QG and further references in - Chap. 9, "Gauge Invariant Renormalizability of Quantum Gravity".

积分空间 g^i 包含轨道，即由度规场的规范变换 (10) 定义的子空间。由于作用量在任意这类子空间上都保持恒定，显然每个子空间对积分的贡献都是无穷大，因此该积分是不良定义的。这种发散类似于对退化矩阵的行列式取对数。该发散是规范不变性的直接结果，正是引力和杨-米尔斯理论中德维特-法捷耶夫-波波夫方法 [13, 14] 所要解决的问题 (也可参见例如 [15] 中关于量子引力的讨论，以及第 9 章“量子引力的规范不变可重整化性”中的更多参考文献)。

The first step is to insert the factor of unity in the integrand of (11), in the form

第一步是在 (11) 式的被积函数中插入单位因子，形式如下

$$1 = \Delta(g) \int d\xi^\alpha \delta\left(\chi^\alpha(g) - l^\alpha\right). \tag{12}$$

Here l^α is an arbitrary vector field and $\chi^\alpha(g)$ is the gauge-fixing condition. There can be different choices of this condition but, in principle, one can think that the surface $\chi^\alpha(g) = l^\alpha$ (or, simply, $\chi^\alpha(g) = 0$) crosses each orbit in a unique "point," such that the degeneracy is removed. Indeed, this is not the way the Faddeev-Popov method works, as we will see.

此处 l^α 是任意矢量场，$\chi^\alpha(g)$ 是规范固定条件。该条件有多种不同选择，但原则上可以认为曲面 $\chi^\alpha(g) = l^\alpha$ (或简写为 $\chi^\alpha(g) = 0$) 与每条轨道仅相交于唯一一个“点”，从而消除简并。但正如我们之后会看到的，法捷耶夫-波波夫方法实际上并非如此运作。

The integral (12) can be easily taken by making the change of integration variables to χ^α ,

将积分变量换为 χ^α 后，可以很容易地算出积分 (12),

$$\Delta^{-1}(g) = \int d\chi^\beta \,\mathrm{Det}\left(\frac{\delta\xi^\alpha}{\delta\chi^\beta}\right) \delta\left(\chi^\alpha(g) - l^\alpha\right). \tag{13}$$

The Jacobian of this transformation is inverse to the matrix

该变换的雅可比行列式是如下矩阵的逆

$$\left(\frac{\delta\xi^\alpha}{\delta\chi^\beta}\right)^{-1} = \frac{\delta\chi_\alpha}{\delta\xi_\beta} = \frac{\delta\chi_\alpha}{\delta g^i} R^{i\beta} = \frac{\delta\chi_\alpha}{\delta g_{\mu\nu}} R_{\mu\nu}{}^{\beta} = M_\alpha{}^\beta. \tag{14}$$

It is important that the determinant of the matrix $M_\alpha{}^\beta$ does not depend on ξ , but only on g^i and the form of the gauge-fixing condition $\chi(g)$. In this way, we arrive at the equivalent, albeit already nondegenerate, form of (11),

需要注意的是，矩阵 $M_\alpha{}^\beta$ 的行列式不依赖于 ξ ，仅依赖于 g^i 和规范固定条件 $\chi(g)$ 的形式。通过这种方式，我们得到了 (11) 式等价的非简并形式:

$$Z = \int dg\delta\left(\chi^{\alpha}(g) - l^{\alpha}\right)\mathrm{Det}\left(M_{\alpha}^{\beta}\right)e^{iS(g)}. \tag{15}$$

To achieve a more useful form of this expression, we insert in the integrand a unit in the form

为了得到这个表达式更实用的形式，我们在被积函数中插入如下形式的单位

$$1 = \left(\mathrm{Det}\, Y_{\alpha\beta}\right)^{1/2}\int dl^{\alpha}\exp\left\{\frac{i}{2}l^{\alpha}Y_{\alpha\beta}l^{\beta}\right\}. \tag{16}$$

Here $Y_{\alpha\beta}$ is a new object identified as weight operator. We will discuss several forms of this operator, adapted to different models of QG, in what follows.

此处 $Y_{\alpha\beta}$ 是一个被定义为权算符的新对象。后续我们将讨论适配不同量子引力模型的多种权算符形式。

Taking the integral with the delta function, we get

对含 δ 函数的部分积分后，我们得到

$$Z = \int dg\left(\mathrm{Det}\, Y_{\alpha\beta}\right)^{1/2}\left(\mathrm{Det}\, M_{\alpha}^{\beta}\right)e^{iS+iS_{gf}}, \tag{17}$$

where

其中

$$S_{gf} = \frac{1}{2}\chi^{\alpha}Y_{\alpha\beta}\chi^{\beta}. \tag{18}$$

In some situations, it is useful to deal with the modified form of the operator,

在一些情形下，处理该算符的修改形式会更方便，

$$\widetilde{M}_{\alpha\beta} = Y_{\alpha\lambda}M_{\lambda}^{\beta}, \tag{19}$$

providing an alternative form of (17), i.e.,

给出式 (17) 的另一种形式，即

$$\begin{aligned} Z &= \int dg\left(\mathrm{Det}\, Y_{\alpha\beta}\right)^{-1/2}\left(\mathrm{Det}\, \widetilde{M}_{\alpha}^{\beta}\right)e^{iS+iS_{gf}} \\ &= \int dgd\bar{C}dC\left(\mathrm{Det}\, Y_{\alpha\beta}\right)^{-1/2}e^{iS+iS_{gf}+iS_{gh}}, \end{aligned} \tag{20}$$

where the action of the Faddeev-Popov ghosts has the form

其中法捷耶夫-波波鬼场的作用量形如

$$S_{gh} = \bar{C}^{\alpha}\widetilde{M}_{\alpha}^{\beta}C_{\beta}. \tag{21}$$

It is clear that the ghost fields $\bar{C}^{\alpha}$ and C_{β} should be fermions (i.e., have odd Grassmann parity) to provide the positive power of Det $\widetilde{M}_{\alpha}{}^{\beta}$. One can trade the functional determinant $\left(\text{Det}\, Y_{\alpha\beta}\right)^{-1/2}$ for another (third) ghost. This ghost may have even or odd Grassmann parity, respectively, for the versions with $\widetilde{M}_{\alpha\beta}$ or $M_{\lambda}{}^{\beta}$.

显然鬼场 $\bar{C}^{\alpha}$ 和 C_{β} 应为费米子 (即具有奇数格拉斯曼奇偶性)，以保证 Det $\widetilde{M}_{\alpha}{}^{\beta}$ 为正幂次。我们可以将函数行列式 $\left(\text{Det}\, Y_{\alpha\beta}\right)^{-1/2}$ 替换为另一个 (第三) 鬼场。对于带 $\widetilde{M}_{\alpha\beta}$ 或 $M_{\lambda}{}^{\beta}$ 的版本，该鬼场分别具有偶数或奇数格拉斯曼奇偶性。

Thus, the total Faddeev-Popov action $S_t = S + S_{gf} + S_{gh}$ depends on the choice of the gauge-fixing condition χ^{α} and the weight operator (sometimes called weight function) $Y_{\alpha\beta}$. In what follows, we consider a few important examples of the choice of these two objects in different models of QG. As with most of this review, the reader can find more detailed discussion in [7].

因此，总法捷耶夫-波波作用量 $S_t = S + S_{gf} + S_{gh}$ 依赖于规范固定条件 χ^{α} 与权重算符 (有时称为权重函数) $Y_{\alpha\beta}$ 的选取。下文我们将讨论不同量子引力模型中这两个对象选取的几个重要例子。和本综述大部分内容一样，读者可在文献 [7] 中找到更详细的讨论。

Quantum General Relativity

量子广义相对论

The first and most obvious candidate for the starting action of QG is the Einstein-Hilbert functional

量子引力 (QG) 作用量最首要且最显而易见的候选就是爱因斯坦-希尔伯特泛函

$$S_{EH} = -\frac{1}{\kappa^2}\int d^4x\sqrt{-g}\,(R + 2\Lambda)\,. \tag{22}$$

Here we denoted $\kappa^2 = 16\pi G$, as it proves useful when we consider the perturbative expansion. The reason is that, modifying the expansion (9) to

此处我们标记为 $\kappa^2 = 16\pi G$ ，这在考虑微扰展开时非常有用。原因是，将展开式 (9) 修改为

$$g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \kappa h_{\mu\nu}, \tag{23}$$

the bilinear in $h_{\mu\nu}$ terms are free from the parameter κ , while the higher orders in $h_{\mu\nu}$ terms have positive powers κ . One of the consequences is that κ plays the role of the interaction constant in this QG model. In quantum theory, κ turns out the loop expansion parameter. Let us stress that this is, typically, not so for other QG models, which may have other parameters of the loop expansion. Other two relevant observations is that, since gravity is a non-polynomial theory, the parametrization (23) results in that the action (22) has unbounded powers of κ . Furthermore, things get more complicated in more general parameterizations but, typically, κ remains the parameter of the loop expansion.

$h_{\mu\nu}$ 的二次项不含参数 κ，而 $h_{\mu\nu}$ 的高阶项包含正幂次的 κ。由此得到的一个结论是，κ 在该量子引力模型中充当相互作用常数。在量子理论中，κ 恰好是圈展开参数。需要强调的是，其他量子引力模型通常并非如此，它们可能拥有其他的圈展开参数。另外两个相关结论是: 由于引力是一个非多项式理论，参数化 (23) 使得作用量 (22) 包含无界幂次的 κ。此外，在更一般的参数化下情况会更复杂，但 κ 通常仍是圈展开参数。

The flat limit requires a vanishing cosmological constant and we get

平直极限要求宇宙常数为零，我们得到

$$g_{\mu\upsilon} = \eta_{\mu\upsilon} + \kappa h_{\mu\upsilon}. \tag{24}$$

As we shall see below, this means that the propagator of quantum metric is free of κ by construction.

如下文所示，这说明按构造量子度规的传播子本身就不含 κ。

Instead of the expansion of the metric $g_{\mu\upsilon}$, one can start from the expansion of the inverse metric $g^{\mu\upsilon}$, or even a more general parametrization of the quantum metric. Later on, we consider the version which is the most general for the one-loop calculations and has various arbitrary parameters.

除了对度规做 $g_{\mu\upsilon}$ 展开，也可以从逆度量 $g^{\mu\upsilon}$ 展开出发，甚至可以对量子度规采用更一般的参数化。后续我们会讨论适用于单圈计算的最通用形式，它包含多个任意参数。

The Faddeev-Popov procedure requires introducing the gauge-fixing term S_{gf}. Such a term is called to make the highest derivative in $h_{\mu\upsilon}$ terms of the action $S + S_{gf}$ nondegenerate. Then one can obtain the propagator of this field or, e.g., apply the heat-kernel method and Schwinger-DeWitt technique for calculating the divergences in a covariant form. Since the theory (22) has at most second derivatives, we can choose the weight operator proportional to the background metric and arrive at the gauge-fixing term in the form

法捷耶夫-波波夫程序需要引入规范固定项 S_{gf}。引入该项是为了让作用量 $S + S_{gf}$ 中 $h_{\mu\upsilon}$ 项的最高导数非退化。之后我们就可以得到该场的传播子，或是应用热核方法与施温格-德维特技术计算协变形式下的发散。由于理论 (22) 最多只包含二阶导数，我们可以选择权重算符与背景度规成正比，最终得到如下形式的规范固定项

$$S_{gf} = \frac{1}{\alpha}\int d^4x\sqrt{-g}\chi_\mu\chi^\mu, \ \text{where} \ \chi_\mu = \nabla_\upsilon h^\upsilon_\mu - \beta\nabla_\mu h \tag{25}$$

and α and β are the gauge-fixing parameters. The dependence on the choice of the gauge fixing and on the parametrization of quantum metric represents an important part of the QG development.

其中 α 和 β 是规范固定参数。对规范固定选择的依赖，以及对量子度规参数化的依赖，是量子引力发展过程的重要组成部分。

Why does the action (22) require the simplest weight functional in the case of QG? The reason is that the Lagrangian of GR (22) has at most two derivatives of the quantum metric $h_{\mu\upsilon}$ in the action. Thus, a ”correct” (or, better to say, appropriate) way of breaking down the degeneracy in the total action requires that the gauge-fixing term S_{gf} also has two derivatives. Since each of χ_μ in Eq. (25) is linear in derivatives, we are forced to implement the choice $Y_{\mu\nu} = \text{const} \times g_{\mu\nu}$ in this case. With some adjustments, the same logic will be used in all models of QG which will be discussed below (and even those which will not be).

为什么作用量 (22) 在量子引力情形下要求最简单的权重泛函? 原因是广义相对论的拉格朗日量 (22) 在作用量中最多包含量子度规 $h_{\mu\nu}$ 的二阶导数。因此，打破总作用量简并的「正确」(或更准确地说，合适) 方法要求规范固定项 S_{gf} 同样也包含二阶导数。由于式 (25) 中每个 χ_μ 对导数都是线性的，这种情况下我们不得不取 $Y_{\mu\nu} =$ 为常数 $\times g_{\mu\nu}$ 。经过适当调整，该逻辑也适用于下文将要讨论 (甚至那些未讨论) 的所有量子引力模型。

The action of ghosts is constructed in a standard way, as

鬼场的作用量按标准方式构造，如下

$$S_{gh} = \int d^4x \sqrt{-g} \bar{C}^\alpha M_\alpha^\beta C_\beta, \tag{26}$$

where the operator M is the variation of the gauge-fixing condition with respect to the transformation function,

其中算符 M 是规范固定条件对变换函数的变分,

$$M_\alpha^\beta = \frac{\delta \chi_\alpha}{\delta \xi_\beta} = \frac{\delta \chi_\alpha}{\delta h_{\mu\nu}} R_{\mu\nu}{}^\beta. \tag{27}$$

It is important that the ghost fields in (26) satisfy the second-order equations. This means the propagators of both gravitational field $h_{\mu\nu}$ and ghosts have the same type of UV behavior $1/k^2$. In the next models, we shall see that providing the homogeneity of the propagators of different modes of the metric (8) and of the ghosts may require some extra efforts. And such a homogeneity worth these efforts, as without it, the quantum theory gains a lot of artificial complications.

重要的一点是，(26) 中的鬼场满足二阶方程。这意味着引力场 $h_{\mu\nu}$ 和鬼场的传播子具有相同类型的紫外行为 $1/k^2$ 。在后续模型中我们会看到，要保证度规不同模式 (8) 和鬼场传播子齐性往往需要额外处理，而这种齐性是值得付出这些努力的，因为如果没有它，量子理论会产生大量人为引入的复杂性。

Fourth-Derivative Gravity

四阶导数引力

The next model of common interest is based on the fourth-derivative action. If the previous choice, namely, the Einstein-Hilbert action of GR, is strongly motivated by the success of Einstein's classical gravitational theory, the fourth-derivative model is motivated by the consistency conditions of semiclassical gravity. It is known from the early paper by Utiyama and DeWitt [16] (see also the aforementioned books [2, 3, 5, 7]) that if the matter fields are quantum, the action of vacuum (i.e., the gravitational action) of the renormalizable theory has to include both the Einstein-Hilbert action (22) and the covariant local fourth-derivative terms

下一个广受关注的模型基于四阶导数作用量。如果说此前的选择，也就是广义相对论的爱因斯坦-希尔伯特作用量，是由爱因斯坦经典引力理论的成功强力推动的，那么四阶导数模型的动机则来自半经典引力的一致性条件。从 Utiyama 和 DeWitt 的早期论文 [16](也参见前述书籍 [2, 3, 5, 7])) 可知:如果物质场是量子化的，可重整化理论的真空作用量 (即引力作用量) 必须同时包含爱因斯坦-希尔伯特作用量 (22) 和协变局部四阶导数项

$$S_{HD} = \int d^4x\sqrt{-g}\left\{a_1 R^2_{\mu\nu\alpha\beta} + a_2 R^2_{\mu\nu} + a_3 R^2 + a_4 \Box R\right\}. \tag{28}$$

In QG, it is more useful to write this action in another basis, including the square of the Weyl tensor and the integrand of the Gauss-Bonnet topological term

在量子引力中，将该作用量写在另一组基下更为实用，这组基包含外尔张量的平方和高斯-博内拓扑项的被积函数

$$C^2 = C_{\mu\upsilon\alpha\beta}C^{\mu\upsilon\alpha\beta} = R^2_{\mu\upsilon\alpha\beta} - 2R^2_{\mu\upsilon} + \frac{1}{3}R^2,$$

$$E_4 = R^2_{\mu\upsilon\alpha\beta} - 4R^2_{\mu\upsilon} + R^2 \tag{29}$$

In this basis, the fourth-derivative action has the form

在这组基下，四阶导数作用量形式为

$$S_{HD} = -\int d^4x\sqrt{-g}\left\{\frac{1}{2\lambda}C^2 - \frac{1}{\rho}E_4 + \frac{1}{\xi}R^2 + \tau\Box R + \frac{1}{\kappa^2}(R - 2\Lambda)\right\}. \tag{30}$$

Sometimes other notations for the couplings ρ and ξ are used, e.g., $\theta = \lambda/\rho$ and $\omega = -3\lambda/\xi$, or the ones used in Chap. 11,"One-Loop Divergences in Higher-Derivative Gravity" about the one-loop calculations and renormalization group flows in the theory (28). The important part is the positive sign of the coupling λ , as it is required by the positivity of the energy of the massless tensor mode (graviton) in the high-energy region [17, 18] (see also an alternative treatment of the same problem in [7]).

有时耦合常数 ρ 和 ξ 会使用其他记号，例如 $\theta = \lambda/\rho$ 和 $\omega = -3\lambda/\xi$ ，或是第 11 章“高阶导数引力中的单圈发散”所用的记号，该章讨论了理论 (28) 中的单圈计算和重整化群流。其中重要的一点是耦合常数 λ 符号为正，这是高能区无质量张量模 (引力子) 能量为正的要求 [17, 18](另见文献 [7] 对同一问题的不同处理)。

Introducing the gauge-fixing term in the fourth-derivative theory of QG is a more complicated task compared to the quantum GR. Since we are interested to maintain homogeneity of the propagator, the gauge-fixing term should have four derivatives of the quantum metric, and hence, we introduce the expression for the gauge-fixing action

与量子广义相对论相比，在四阶导数量子引力理论中引入规范固定项要更复杂。由于我们希望保持传播子的齐次性，规范固定项应当包含量子度规的四阶导数，因此我们给出规范固定作用量的表达式

$$S_{gf} = \frac{1}{2} \int d^4x \sqrt{-g} \chi^\mu Y_{\mu\upsilon} \chi^\upsilon, \tag{31}$$

where the gauge condition χ^μ is still defined by the formula (25), but the new weight function $Y_{\mu\nu}$ should be a nondegenerate operator of the second order in derivatives. In the framework of the background field method, its most general form is

其中规范条件 χ^μ 仍由式 (25) 定义，但新的权重函数 $Y_{\mu\nu}$ 必须是二阶导数的非退化算符。在背景场方法的框架下，它的最一般形式为

$$Y_{\mu\upsilon} = \frac{1}{\alpha} \left(g_{\mu\upsilon} \Box + \gamma \nabla_\mu \nabla_\upsilon - \nabla_\upsilon \nabla_\mu + p_1 R_{\mu\upsilon} + p_2 R g_{\mu\upsilon} \right), \tag{32}$$

where the nondegeneracy requires $\gamma \neq 0$. Compared to the quantum GR, the new gauge-fixing condition depends on a larger number of arbitrary gauge-fixing parameters, i.e., on $\alpha_i = (\alpha, \beta, \gamma, p_1, p_2)$, where β comes from Eq. (25). In the case of the flat background metric, $Y_{\mu\nu}$ has only two arbitrary parameters α and γ .

其中非退化性要求 $\gamma \neq 0$ 。与量子广义相对论相比，新的规范固定条件依赖更多个任意规范固定参数，即依赖 $\alpha_i = (\alpha, \beta, \gamma, p_1, p_2)$ ，其中 β 来自式 (25)。在平坦背景度规的情况下，$Y_{\mu\nu}$ 仅存在两个任意参数 α 和 γ 。

With the definition (31) and (32), all the modes of the quantum metric in (8) have the same leading UV behavior of the propagator, $G_i^{-1}(k) \propto k^4$. As we shall see in the forthcoming sections, the power counting is greatly simplified if the ghost action has the same number of derivatives as the action of the $h_{\mu\nu}$ field. It is clear that the problem can be solved by introducing a modified ghost action (21), namely,

通过定义 (31) 和 (32)，(8) 中量子度规的所有模的传播子都具有相同的紫外主导行为 $G_i^{-1}(k) \propto k^4$ 。正如我们在后续章节将会看到的，如果鬼场作用量和 $h_{\mu\nu}$ 场作用量所含导数的数量相同，幂次数计数会大大简化。很明显，我们可以通过引入修改后的鬼场作用量 (21) 解决这个问题，即

$$S_{gh} = \int d^4x \sqrt{-g} \bar{C}^\alpha \widetilde{M}_\alpha{}^\beta C_\beta, \quad \text{where} \quad \widetilde{M}_\alpha{}^\beta = M_\alpha{}^\lambda(g) \, Y_\lambda{}^\beta(g) \,. \tag{33}$$

The functional integral is defined by the general expression (20) with the specific choice (32) of the weight operator. Since both $M_\alpha{}^\lambda$ and $Y_\lambda{}^\beta$ are second-order operators, the propagator of the Faddeev-Popov ghosts behaves like k^{-4} in the UV, exactly as the propagator of gravitational perturbations. This feature proves useful for evaluating the power counting of the Feynman diagrams in this theory.

泛函积分由通式 (20) 定义，并对权重算符做 (32) 的特殊选取。由于 $M_\alpha{}^\lambda$ 和 $Y_\lambda{}^\beta$ 都是二阶算符，法捷耶夫-波波夫鬼的传播子在紫外的行为和引力微扰传播子完全一致，都是 k^{-4} 。这一性质对该理论费曼图的幂次数计数十分有用。

Quantum Gravity Model Polynomial in Derivatives

导数多项式形式的量子引力模型

The previous two examples of QG models are minimal versions. In particular, GR fits all observational and experimental tests for a classical gravity [19] and, in this sense, can be regarded as a reference theory. On the contrary, there is not a single experimental test for the fourth-derivative model of gravity but, from another side, fourth-derivative terms are required for the renormalizability of semiclassical gravity. And the same model guarantees also the renormalizability of QG [17]. On the other hand, the fourth-derivative theory has serious problems related to nonphysical ghosts (which have nothing to do with the Faddeev-Popov ghosts, the traditional use of the same word here is a mere coincidence) and stability, as we shall discuss below. In this situation, one may look beyond the minimal theories and it is natural to try the models with more than four derivatives. Then, the number of derivatives may be finite or infinite. In the first case we meet the polynomial in the derivative models of QG, suggested in [20].

前面两个量子引力 (QG) 模型例子都是最小版本。具体来说，广义相对论 (GR) 通过了经典引力的所有观测和实验检验 [19]，从这个角度来说可以视为参考理论。与之相反，四阶导数引力模型目前还没有任何实验检验，但另一方面，半经典引力的可重整化性要求引入四阶导数项，并且该模型也能保证量子引力的可重整化性 [17]。但另一方面，四阶导数理论存在非物理鬼场相关的严重问题 (这类鬼场和法捷耶夫-波波夫鬼场无关，只是术语巧合) 和稳定性问题，我们会在下文讨论。在这种情况下，我们可以跳出最小理论，自然地尝试导数阶数高于四的模型。此时导数的数量可以是有限的，也可以是无限的。对于导数数量有限的情况，我们就得到了文献 [20] 提出的导数多项式形式量子引力模型。

To construct the polynomial models, we impose the condition that the highest-derivative terms in the action should be homogeneous in the derivatives. In the next section, we shall see that this homogeneity may provide the superrenormalizability of the theory. The action of the theory has the form

构造多项式模型时，我们要求作用量中的最高阶导数项满足导数齐次性条件。在下一节我们会看到，这种齐次性可以让理论具备超可重整化性。该理论的作用量形式为

$$S_N = \int d^4x\sqrt{-g}\,\{\vartheta_{N,R}R\Box^N R + \vartheta_{N,C}C\Box^N C + \vartheta_{N,\mathrm{GB}}\mathrm{GB}_N$$

$$+\vartheta_{N-1,R}R\Box^{N-1}R + \vartheta_{N-1,C}C\Box^{N-1}C + \vartheta_{N-1,\mathrm{GB}}\mathrm{GB}_{N-1} + \ldots$$

$$+\vartheta_{0,R}R^2 + \vartheta_{0,C}C^2 + \vartheta_{0,\mathrm{GB}}\mathrm{GB}_0 + \vartheta_{\mathrm{EH}}R + \vartheta_{\mathrm{cc}} + \mathcal{O}\left(R^3_{\ldots}\right)\}, \tag{34}$$

where $N = 1, 2, \ldots$ and all ϑ ’ s are arbitrary parameters of the action. It is assumed that both $\vartheta_{N,R}$ and $\vartheta_{N,C}$ are nonzero and that the maximal power of metric derivatives in the terms $\mathcal{O}\left(R^3\right)$ is $2N + 4$, i.e., is not higher than of the terms of the second order in curvatures, i.e., $\mathcal{O}\left(R^2\right)$.

其中 $N = 1, 2, \ldots$ 和所有 ϑ 都是作用量的任意参数。假设 $\vartheta_{N,R}$ 和 $\vartheta_{N,C}$ 都不为零，项 $\mathcal{O}\left(R^3\right)$ 中度量导数的最高幂次不超过曲率二阶项的导数最高幂次，即不超过 $2N + 4$ ，也就是满足 $\mathcal{O}\left(R^2\right)$ 。

Furthermore, there are the squares of the Weyl tensor with extra factors of $\Box$,

此外，还有带额外因子 $\Box$ 的外尔张量平方项，

$$C\Box^n C = C_{\mu\upsilon\alpha\beta}\Box^n C^{\mu\upsilon\alpha\beta} = R_{\mu\upsilon\alpha\beta}\Box^n R^{\mu\upsilon\alpha\beta} - 2R_{\mu\upsilon}\Box^n R^{\mu\upsilon} + \frac{1}{3}R\Box^n R. \tag{3,5}$$

Similarly, using integrations by parts and the Bianchi identities, the generalized Gauss-Bonnet invariants can be shown to have the property

类似地，通过分部积分和比安基恒等式，可以证明广义高斯-博内不变量满足如下性质

$$\mathrm{GB}_n = R_{\mu\upsilon\alpha\beta}\Box^n R^{\mu\upsilon\alpha\beta} - 4R_{\mu\upsilon}\Box^n R^{\mu\upsilon} + R\Box^n R = \mathcal{O}\left(R^3_{\dots}\right). \tag{36}$$

This term is not topological for $n \geq 1$, but it contributes only to the third- and higher-order terms in the curvature tensor and to the total derivatives. Thus, it may affect the vertices but not the propagators of QG, as we shall explicitly check out in what follows. On a flat background, the $\mathcal{O}\left(R^2\right)$ terms may contribute to the propagator of the gravitational perturbation, while $\mathcal{O}\left(R^3\right)$ terms affect only the vertices. In the action (34), the $\mathcal{O}\left(R^2_{\dots}\right)$ s are given on the basis of Weyl-squared and R -squared terms. As will be shown below, the Weyl-squared terms affect the propagation of the tensor mode $\bar{h}^{\perp\perp}_{\mu\upsilon}$, and the R -squared $R \cdot R$ terms affect the propagation of the scalar modes. Using the higher-derivative actions in the form (34), we separate the propagators of the tensor and scalar modes at the level of the action.

当 $n \geq 1$ 时，该项不是拓扑项，但它仅对曲率张量的三阶及更高阶项和全导数有贡献。因此正如我们后续会明确验证的，它会影响顶点，但不影响量子引力的传播子。在平坦背景上，$\mathcal{O}\left(R^2\right)$ 项可以对引力微扰的传播子产生贡献，而 $\mathcal{O}\left(R^3\right)$ 项仅影响顶点。在作用量 (34) 中，$\mathcal{O}\left(R^2_{\dots}\right)$ 由外尔平方项和 R 平方项给出。下文会证明，外尔平方项影响张量模式 $\bar{h}^{\perp\perp}_{\mu\upsilon}$ 的传播，而 R 平方 $R \cdot R$ 项影响标量模式的传播。利用形式为 (34) 的高阶导数作用量，我们可以在作用量层面分离张量模式和标量模式的传播子。

To provide the homogeneity in derivatives for the propagator of all modes (8), the gauge-fixing terms should have the same highest power in derivatives as the main action. Since the gauge-fixing conditions χ_μ are always chosen in the form (25), the $\mathcal{O}\left(p^{-4-2N}\right)$ propagator of the quantum metric requires the weight function to be

为了让所有模式传播子 (8) 都满足导数齐次性，规范固定项的最高导数幂次需要和主作用量保持一致。由于规范固定条件 χ_μ 始终选取为形式 (25)，量子度量的 $\mathcal{O}\left(p^{-4-2N}\right)$ 传播子要求权函数取为

$$Y_{\mu\upsilon} = -\frac{1}{\alpha}\left(g_{\mu\upsilon}\Box + \gamma\nabla_\mu\nabla_\upsilon - \nabla_\upsilon\nabla_\mu\right)\Box^{N+1}. \tag{37}$$

One can add here many terms with a lower order of derivatives, but this does not change the UV behavior of the propagator.

我们可以在这里添加很多低阶导数项，但这不会改变传播子的紫外行为。

To provide the same power of derivatives in the ghost sector, as in the quadratic in curvature action, one can redefine the ghost action as (33), this time with the weight operator (37).

为了让鬼场 sector 的导数幂次和曲率二次作用量中的导数幂次保持一致，我们可以将鬼场作用量重新定义为 (33) 的形式，只需将权算子替换为 (37) 即可。

Furthermore, one can write the action (34) in an alternative form,

此外，我们还可以将作用量 (34) 写成另一种形式，

$$S_N = \int d^4x\sqrt{-g}\Big\{\frac{1}{2}C_{\mu\nu\alpha\beta}P_1\left(\square\right)C^{\mu\nu\alpha\beta} + \frac{1}{2}RP_2\left(\square\right)R$$

$$+\vartheta_{\mathrm{EH}}R + \vartheta_{\mathrm{cc}} + \mathcal{O}\left(R^3_{\ldots}\right)\Big\} \tag{38}$$

where $P_{1,2}\left(x\right)$ are polynomials of the same order N and the terms $\mathcal{O}\left(R^3\right)$ have at most $4+2N$ derivatives of the metric. One can make further generalization of (38), by trading the polynomials to the infinite series of $\square$. The discussion of these theories can be found in the corresponding section of the present handbook. Let us just quote the expression for the general (i.e., polynomial or non-polynomial) action

其中 $P_{1,2}\left(x\right)$ 是同阶 N 多项式，项 $\mathcal{O}\left(R^3\right)$ 最多包含度规的 $4+2N$ 阶导数。我们可以进一步推广 (38)，将多项式替换为 $\square$ 的无穷级数。本手册对应章节已有这些理论的讨论，此处仅给出一般 (即多项式或非多项式) 作用量的表达式

$$S_{\text{gen}} = \int d^4x\sqrt{-g}\Big\{-\frac{1}{\kappa^2}\left(R+2\Lambda\right) + \frac{1}{2}C_{\mu\nu\alpha\beta}\Phi\left(\square\right)C_{\mu\nu\alpha\beta}$$

$$+\frac{1}{2}R\Psi\left(\square\right)R + \mathcal{O}\left(R^3_{\ldots}\right)\Big\}. \tag{39}$$

To complete the bilinear in the curvature part of the action, we can add the third higher-derivative term, which boils down to the Gauss-Bonnet topological term for a constant form factor Ω ,

为补全作用量中曲率部分的双线性项，我们可以补充第三项高阶导数项；当形状因子 Ω 为常数时，该项退化为高斯-博内拓扑项，

$$S_{GB} = \frac{1}{2}\int d^4x\sqrt{-g}\left\{R_{\mu\nu\alpha\beta}\Omega\left(\square\right)R_{\mu\nu\alpha\beta} - 4R_{\mu\nu}\Omega\left(\square\right)R_{\mu\nu} + R\Omega\left(\square\right)R\right\}. \tag{40}$$

Despite this term being equivalent to $\mathcal{O}\left(R^3\right)$ in (38), it makes sense to verify this feature, at least at the level of the propagator. The homogeneity of the propagator of all modes of the metric perturbations (8) requires the functions $\Phi\left(x\right)$ and $\Psi\left(x\right)$ to have an analogous behavior in the UV. This can be achieved by requiring that

尽管该项等价于 (38) 中的 $\mathcal{O}\left(R^3\right)$，验证这一性质仍是有意义的，至少在传播子层面需要验证。度规扰动所有模式的传播子 (8) 的齐次性要求函数 $\Phi\left(x\right)$ 和 $\Psi\left(x\right)$ 在紫外区具有相似行为，这可以通过要求

$$\lim_{x\to\infty}\frac{\Psi\left(x\right)}{\Phi\left(x\right)} = C, \tag{41}$$

with C being a nonzero constant. For the polynomial functions $P_{1,2}\left(x\right)$ of the same order, in (38), the last condition is guaranteed. In the case of the non-polynomial functions, the simplest useful choice is

其中 C 为非零常数。对于同阶多项式函数 $P_{1,2}(x)$，(38) 式已经保证了最后这个条件。对于非多项式函数，最简单的常用选择是

$$\Phi(x) = -\frac{1}{\kappa^2 x}\left(e^{\alpha_1 x} - 1\right) \text{ and } \Psi(x) = -\frac{C}{\kappa^2 x} e^{\alpha_2 x}, \tag{42}$$

such that condition (41) reduces to $\alpha_1 = \alpha_2$.

使得条件 (41) 约化为 $\alpha_1 = \alpha_2$ 。

The gauge-fixing term in theory (39) that provides the homogeneity of the propagators is of the standard form (31), but requires the special weight operator

理论 (39) 中保证传播子齐次性的规范固定项是标准形式 (31)，但需要特殊的权算符

$$Y_{\mu\nu} = \left(g_{\mu\nu}\Box - \gamma\nabla_\mu\nabla_\nu + p_1 R_{\mu\nu} + p_2 R g_{\mu\nu}\right) W(\Box)\,. \tag{43}$$

In many cases, it is sufficient to consider $W(\Box) \propto \Phi(\Box)$, but we shall keep this function arbitrary for generality, until some point.

很多情况下，取 $W(\Box) \propto \Phi(\Box)$ 就足够了，但为了保证一般性，在此之前我们保留该函数的任意性。

Finally, the homogeneity in the momentum at the UV for the quantum metric and for the Faddeev-Popov ghosts can be achieved by the standard replacement (33).

最后，量子度规和法捷耶夫-波波夫鬼场在紫外区动量的齐次性，可以通过标准替换 (33) 实现。

Bilinear Forms and Linear Approximation

双线性型与线性近似

The analysis of propagators in different models of quantum gravity requires the bilinear expansions of the relevant quantities depending on the curvature tensor, on a flat metric background. On the other hand, similar expressions with an arbitrary background metric are useful for the one-loop calculations in the background field method. Thus, we consider a general case and assume the expansion (9), i.e. $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$.

分析不同量子引力模型中的传播子时，需要对依赖曲率张量的相关量在平直度规背景下做双线性展开。另一方面，含任意背景度规的类似表达式可用于背景场方法中的单圈计算。因此我们考虑一般情形，采用展开式 (9)，即 $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$.

Let us refer the interested reader to the book [7] for technical details and only give the following list of basic expansions:

感兴趣的读者可查阅文献 [7] 了解技术细节，此处仅列出如下基本展开式:

$$g'^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + h^{\mu}{}_{\lambda} h^{\upsilon\lambda} - h^{\mu}_{\lambda} h^{\lambda}_{\tau} h^{\upsilon\tau} + \ldots$$

$$\sqrt{-g'} = \sqrt{-g}\left(1 + \frac{1}{2}h + \frac{1}{8}h^2 - \frac{1}{4}h_{\mu\nu}h^{\mu\nu} + \ldots\right),$$

$$\Gamma'^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \delta\Gamma^{\alpha}_{\beta\gamma}, \tag{44}$$

where

其中

$$\delta\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}\left(g^{\alpha\lambda} - h^{\alpha\lambda} + h^{\alpha}_{\kappa}h^{\lambda\kappa} - h^{\alpha}_{\kappa}h^{\kappa}_{\tau}h^{\tau\lambda} + \ldots\right)\left(\nabla_{\beta}h_{\gamma\lambda} + \nabla_{\gamma}h_{\beta\lambda} - \nabla_{\lambda}h_{\gamma\beta}\right).$$

In these formulas, the Greek indices are lowered and raised with the background metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$. One has to remember that the variation $\delta\Gamma^{\lambda}_{\mu\nu}$ is a tensor and, therefore, can be a subject to covariant differentiation.

在这些公式中，希腊指标用背景度规 $g_{\mu\nu}$ 和它的逆度规 $g^{\mu\nu}$ 升降。需要注意，变分 $\delta\Gamma^{\lambda}_{\mu\nu}$ 是张量，因此可以进行协变微分。

The first two orders of expansion of the Riemann tensor have the form

黎曼张量的前两阶展开形式为

$$R'^{\alpha}_{\cdot\cdot\beta\mu\nu} = R^{\alpha}_{\cdot\beta\mu\nu} + \delta R^{\alpha}_{\cdot\beta\mu\nu}, \ \text{ where } \ \delta R^{\alpha}_{\cdot\beta\mu\nu} = R^{(1)\alpha}_{\cdot\cdot\beta\mu\nu} + R^{(2)\alpha}_{\cdot\cdot\beta\mu\nu}, \tag{45}$$

$$R^{(1)\alpha}_{\cdot\beta\mu\nu} = \frac{1}{2}\left(\nabla_{\mu}\nabla_{\beta}h^{\alpha}_{\upsilon} - \nabla_{\upsilon}\nabla_{\beta}h^{\alpha}_{\mu} + \nabla_{\upsilon}\nabla^{\alpha}h_{\mu\beta} - \nabla_{\mu}\nabla^{\alpha}h_{\upsilon\beta}\right.$$

$$\left. + R^{\alpha}_{\tau\mu\nu}h^{\tau}_{\beta} - R^{\tau}_{\beta\mu\nu}h_{\tau\alpha}\right)$$

$$R^{(2)\alpha}_{\cdot\beta\mu\nu} = \frac{1}{2}h^{\alpha\lambda}\left\{\nabla_{\mu}\nabla_{\lambda}h_{\nu\beta} - \nabla_{\nu}\nabla_{\lambda}h_{\mu\beta} + \nabla_{\nu}\nabla_{\beta}h_{\mu\lambda} - \nabla_{\mu}\nabla_{\beta}h_{\nu\lambda}\right.$$

$$\left. + \left[\nabla_{\upsilon}, \nabla_{\mu}\right]h_{\beta\lambda}\right\} + \frac{1}{4}\left\{\left(\nabla_{\mu}h^{\alpha\lambda}\right)\left(\nabla_{\lambda}h_{\upsilon\beta} - \nabla_{\beta}h_{\upsilon\lambda} - \nabla_{\upsilon}h_{\lambda\beta}\right)\right.$$

$$-\left(\nabla_{\beta}h^{\lambda}_{\mu} + \nabla_{\mu}h^{\lambda}_{\beta}\right)\left(\nabla_{\lambda}h^{\alpha}_{\upsilon} - \nabla^{\alpha}h_{\lambda\upsilon}\right) + \left(\nabla^{\lambda}h_{\upsilon\beta}\right)\left(\nabla^{\alpha}h_{\mu\lambda} - \nabla_{\lambda}h^{\alpha}_{\mu}\right)$$

$$-\left(\nabla_{\upsilon}h^{\alpha\lambda}\right)\left(\nabla_{\lambda}h_{\mu\beta} - \nabla_{\beta}h_{\mu\lambda} - \nabla_{\mu}h_{\lambda\beta}\right) - \left(\nabla^{\lambda}h_{\mu\beta}\right)\left(\nabla^{\alpha}h_{\upsilon\lambda} - \nabla_{\lambda}h^{\alpha}_{\upsilon}\right)$$

$$\left. + \left(\nabla_{\beta}h^{\lambda}_{\upsilon} + \nabla_{\upsilon}h^{\lambda}_{\beta}\right)\left(\nabla_{\lambda}h^{\alpha}_{\mu} - \nabla^{\alpha}h_{\lambda\mu}\right)\right\}.$$

Here and in what follows, points indicate the positions of the raised indices. Similar formulas for the Ricci tensor and scalar curvature are

下文中，点表示升指标所在的位置。里奇张量和标量曲率的类似公式为

$$R'_{\beta v} = R_{\mu v} + \delta R_{\mu v}, \text{ where } \delta R_{\mu v} = R^{(1)}_{\mu v} + R^{(2)}_{\mu v},$$

$$R^{(1)}_{\mu v} = \frac{1}{2}\left(\nabla_\lambda \nabla_\mu h^\lambda_v + \nabla_\lambda \nabla_v h^\lambda_\mu - \nabla_\mu \nabla_\mu h - \Box h_{\mu v}\right), \tag{46}$$

$$R^{(2)}_{\mu\nu} = \frac{1}{2} h^{\alpha\beta}\left(\nabla_\alpha \nabla_\beta h_{\mu\nu} + \nabla_\mu \nabla_\nu h_{\alpha\beta} - \nabla_\alpha \nabla_\mu h_{\beta\nu} - \nabla_\alpha \nabla_\nu h_{\beta\mu}\right)$$

$$+\frac{1}{2}\left(\nabla_\alpha h_{\mu\beta}\right)\left(\nabla^\alpha h_v{}^\beta - \nabla^\beta h_v{}^\alpha\right) + \frac{1}{4}\left(\nabla_\mu h_{\alpha\beta}\right)\left(\nabla_v h^{\alpha\beta}\right)$$

$$+\frac{1}{4}\left(2\nabla_\beta h^{\alpha\beta} - \nabla^\alpha h\right)\left(\nabla_\alpha h^{\mu v} - \nabla_\mu h^{\alpha v} - \nabla_v h^{\alpha\mu}\right)$$

and $R' = R + \delta R$, where $\delta R = R^{(1)} + R^{(2)}$,

且 $R' = R + \delta R$，其中 $\delta R = R^{(1)} + R^{(2)}$,

(47)

$$R^{(1)} = \nabla_\mu \nabla_\nu h^{\mu\nu} - \Box h - R_{\mu\nu} h^{\mu\nu},$$

$$R^{(2)} = h^{\alpha\beta}\left(\nabla_\alpha \nabla_\beta h + \Box h_{\alpha\beta} - \nabla_\alpha \nabla_\mu h^\mu_\beta - \nabla_\mu \nabla_\alpha h^\mu_\beta\right) - \frac{1}{4}\left(\nabla_\alpha h\right)\left(\nabla^\alpha h\right)$$

$$+\frac{1}{4}\left(\nabla_\mu h_{\alpha\beta}\right)\left(3\nabla^\mu h^{\alpha\beta} - 2\nabla^\alpha h^{\mu\beta}\right) + \left(\nabla_\alpha h^{\alpha\beta}\right)\left(\nabla_\beta h - \nabla_\mu h^\mu{}_\beta\right) + R_{\mu v} h^{\mu\mu}_{\alpha} h^{v\alpha}.$$

Using the expressions listed above, one can easily get the expansions of the terms in the action, up the second order in $h_{\mu\nu}$. The results can be written for the terms in the four-derivative action (30), but it is easy to show how, in the particular case of a flat metric, these expansions can be mapped to the more general action (39) with an arbitrary (finite or even infinite) number of derivatives.

利用上述列出的表达式，可以很容易得到作用量中各项直到 $h_{\mu\nu}$ 二阶的展开式。结果可以写为四导数作用量 (30) 中的项，同时不难看出，在平直度规的特殊情形下，这些展开可以对应到更一般的作用量 (39)，该作用量含有任意 (有限甚至无穷多) 导数项。

The first expansion has the form

第一个展开式形式为

$$\left(\int d^4x\sqrt{-g'}\left[R' + 2\Lambda\right]\right)^{(2)} = \frac{1}{4}\int d^4x\sqrt{-g} h^{\mu v}\left[\delta_{\mu v,\alpha\beta}\Box - g_{\mu v} g_{\alpha\beta}\Box\right.$$

$$-2g_{\mu\alpha}\nabla_v \nabla_\beta + \left(g_{\mu v}\nabla_\alpha \nabla_\beta - g_{\alpha\beta}\nabla_\mu \nabla_v\right) - \left(g_{\mu v} R_{\alpha\beta} - g_{\alpha\beta} R_{\mu v}\right)$$

$$\left.+2R_{\mu\alpha v\beta} - (R + 2\Lambda)\left(\delta_{\mu v,\alpha\beta} - \frac{1}{2} g_{\mu v} g_{\alpha\beta}\right)\right] h^{\alpha\beta},$$

where we used the DeWitt notation for the unit matrix in the symmetric tensor space

其中我们对对称张量空间中的单位矩阵采用了德维特记号

$$\delta_{\mu\upsilon,\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha}g_{\upsilon\beta} + g_{\upsilon\alpha}g_{\mu\beta}\right) \tag{48}$$

and the short notations that assume the symmetrization, e.g.,

且采用了隐含对称化的简写，例如

$$g_{\mu\alpha}\nabla_\upsilon\nabla_\beta \to \frac{1}{4}\left(g_{\mu\alpha}\nabla_\upsilon\nabla_\beta + g_{\upsilon\alpha}\nabla_\mu\nabla_\beta + g_{\mu\beta}\nabla_\upsilon\nabla_\alpha + g_{\upsilon\beta}\nabla_\mu\nabla_\alpha\right). \tag{49}$$

For the sake of brevity, the remaining expansions will be given only for a flat background. The complete expressions can be found, e.g., in [7]. The simplest of the remaining relevant bilinear expansions is

为简洁起见，后续展开仅给出平直背景下的结果。完整表达式可参见文献 [7]。剩余相关双线性展开中最简单的是

$$\left(\int d^4x\sqrt{-g'}R'^2\right)^{(2)}_{\text{flat}} = \int d^4x h^{\mu\upsilon}\left[\eta_{\alpha\beta}\eta_{\mu\upsilon}\Box^2 - \eta_{\mu\upsilon}\Box\partial_\alpha\partial_\beta\right.$$

$$\left.-\eta_{\alpha\beta}\partial_\mu\partial_\nu\Box + \partial_\mu\partial_\nu\partial_\alpha\partial_\beta\right]h^{\alpha\beta}. \tag{50}$$

One can note the absence of the term $\delta_{\mu\nu,\alpha\beta}\Box^2$ in this expression. As a result, the R^2 term does not affect the propagation of the spin-2 mode $\bar{h}^{\perp\perp}_{\mu\nu}$ of the metric perturbation (8).

可以看到该表达式中不存在项 $\delta_{\mu\nu,\alpha\beta}\Box^2$，因此 R^2 项不影响度规扰动 (8) 自旋 2 模式 $\bar{h}^{\perp\perp}_{\mu\nu}$ 的传播。

The next expansions are the one for the square of the Riemann tensor,

下一个展开式是黎曼张量平方的展开,

$$\left(\int d^4x\sqrt{-g'}R'^2{}_{\mu\nu\alpha\beta}\right)^{(2)}_{\text{flat}} = \int d^4x h^{\mu\nu}\left[\delta_{\mu\nu,\alpha\beta}\Box^2 + \partial_\alpha\partial_\beta\partial_\mu\partial_\upsilon\right.$$

$$\left.-2\eta_{\upsilon\beta}\Box\partial_\mu\partial_\alpha\right]h^{\alpha\beta}, \tag{51}$$

and for the square of the Ricci tensor,

以及里奇张量平方的展开,

$$\left(\int d^4x\sqrt{-g'}R'^2_{\mu\nu}\right)^{(2)}_{\text{flat}} = \frac{1}{2}\int d^4x h^{\mu\nu}\left[\frac{1}{2}\left(\delta_{\mu\upsilon,\alpha\beta} + \eta_{\mu\upsilon}\eta_{\alpha\beta}\right)\Box^2 - \eta_{\upsilon\beta}\Box\nabla_\alpha\nabla_\mu\right.$$

$$\left.+\nabla_\alpha\nabla_\mu\nabla_\beta\nabla_\nu - \frac{1}{2}\eta_{\mu\nu}\Box\nabla_\alpha\nabla_\beta - \frac{1}{2}\eta_{\alpha\beta}\Box\nabla_\mu\nabla_\nu\right]h^{\alpha\beta}. \tag{52}$$

The last two expansions (51) and (52) possess the $\delta_{\mu\nu,\alpha\beta}\Box^2$ terms. This means, these two terms contribute to the flat-space propagator of the transverse and traceless mode of the gravitational perturbation $\bar{h}_{\mu\nu}^{\perp\perp}$ in the representation (8).

最后的两个展开式 (51) 和 (52) 均包含 $\delta_{\mu\nu,\alpha\beta}\Box^2$ 项。这意味着，这两项会对表示 (8) 中引力微扰 $\bar{h}_{\mu\nu}^{\perp\perp}$ 的横向无迹模的平直空间传播子产生贡献。

One can note that it would be impossible to have only one of the terms (51) and (52) contributing to the propagation of the spin-2 mode, because (50) does not contribute to this mode and the linear combination of the three terms (29) forms a topological invariant E_4 .

可以注意到，不可能只有 (51) 和 (52) 中的其中一项对自旋 2 模的传播产生贡献，因为 (50) 对该模没有贡献，而三个项 (29) 的线性组合构成拓扑不变量 E_4 。

Gravitational Waves, Quantization, and Gravitons

引力波、量子化与引力子

The gravitational wave is a dynamical classical solution of Einstein's GR without matter sources. This term can be also used for the solutions of the same sort in modified gravity models. However, since in these models the additional modes are typically massive and, therefore, do not propagate for long distances, it is most common to attribute the notion of a gravitational wave to the solutions in GR. Let us note that the models of massive gravity are left beyond the present handbook. The main reason is that, in these models, quantum aspects do not play an important role. Nowadays, gravitational waves represent one of the most successful parts of gravitational physics, both experimental and theoretical. However, in this short section we present only a brief survey of the basic facts concerning the gravitational waves on a flat background in GR.

引力波是无物质源的爱因斯坦广义相对论的动力学经典解，该术语也可用于修正引力模型中的同类解。但由于这类模型中附加模式通常为有质量模式，因此无法长距离传播，所以引力波这一概念最常被用于指代广义相对论中的解。请注意，有质量引力模型不在本手册讨论范围内，主要原因是在这类模型中量子效应并不发挥重要作用。如今，引力波是引力物理中理论和实验研究都最为成功的方向之一。而在本简短章节中，我们仅简要概述广义相对论平直背景下引力波的基本内容。

Gravitational Waves in a Weak-Gravity Regime

弱引力场中的引力波

We start with an action of gravity with $\Lambda = 0$ and use the bilinear expansion (48) on the flat background metric $g_{\mu\nu} = \eta_{\mu\nu}$. To discuss the emission of the gravitational wave (in the simplest case), we also add the action of matter and consider its approximation that provides a linear equation for $h_{\mu\nu}$. In this way, we obtain the action of GR in the linearized regime,

我们从含 $\Lambda = 0$ 的引力作用量出发，在平坦背景度规 $g_{\mu\nu} = \eta_{\mu\nu}$ 上采用双线性展开 (48)。为讨论引力波的辐射 (最简单情形)，我们还会引入物质作用量，并取其近似形式，得到关于 $h_{\mu\upsilon}$ 的线性方程。由此我们得到线性化区域下广义相对论的作用量:

$$S_{\text{total}}^{(\text{ lin })} = -\frac{1}{32\pi G}\int d^4xh^{\mu\upsilon}\left\{\frac{1}{2}\delta_{\mu\upsilon,\alpha\beta}\Box - \frac{1}{2}\eta_{\mu\upsilon}\eta_{\alpha\beta}\Box - \eta_{\mu\alpha}\partial_\upsilon\partial_\beta\right.$$

$$\left.+\frac{1}{2}\left(\eta_{\mu\upsilon}\partial_\alpha\partial_\beta - \eta_{\alpha\beta}\partial_\mu\partial_\upsilon\right)\right\}h^{\alpha\beta} - \frac{1}{2}\int d^4xh^{\mu\upsilon}T_{\mu\upsilon}, \tag{53}$$

where $\Box = \eta^{\mu\upsilon}\partial_\mu\partial_\upsilon$ and $T_{\mu\upsilon}$ is the energy-momentum tensor of matter in flat spacetime background. The equation for metric perturbations has the form

其中 $\Box = \eta^{\mu\upsilon}\partial_\mu\partial_\upsilon$ 和 $T_{\mu\upsilon}$ 是平坦时空背景下物质的能量动量张量。度规微扰满足的方程形式为

$$\left\{\delta_{\mu\upsilon,\alpha\beta}\Box - \eta_{\mu\upsilon}\eta_{\alpha\beta}\Box - 2\eta_{\mu\alpha}\partial_\upsilon\partial_\beta + \left(\eta_{\mu\upsilon}\partial_\alpha\partial_\beta - \eta_{\alpha\beta}\partial_\mu\partial_\upsilon\right)\right\}h^{\alpha\beta} = 16\pi GT_{\mu\upsilon}.$$

(54)

Multiplying both sides of Eq. (54) by the matrix

将式 (54) 两边乘以下述矩阵

$$K^{\mu\upsilon,\rho\sigma} = \delta^{\mu\upsilon,\rho\sigma} - \frac{1}{2}\eta^{\mu\upsilon}\eta^{\rho\sigma}, \tag{55}$$

we arrive at the equation for the modified stress tensor, $S_{\mu\upsilon} = T_{\mu\upsilon} - \frac{1}{2}T_\lambda^\lambda g_{\mu\upsilon}$,

我们得到修正应力张量满足的方程，即 $S_{\mu\upsilon} = T_{\mu\upsilon} - \frac{1}{2}T_\lambda^\lambda g_{\mu\upsilon}$,

$$\partial_\lambda\partial_\upsilon h_\mu^\lambda + \partial_\lambda\partial_\upsilon h_\mu^\lambda - \Box h_{\mu\upsilon} - \partial_\mu\partial_\mu h = 16\pi GS_{\mu\upsilon}. \tag{56}$$

Here $h = h_{\mu\upsilon}\eta^{\mu\upsilon}$. The last equation describes both propagation and emission of the gravitational waves in the linear approximation. This equation has to be supplemented by the gauge transformation (4) $\delta h_{\mu\nu} = -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu$ and requires imposing the gauge-fixing condition, e.g., using the de Donder (also called Fock-de Donder) condition

此处 $h = h_{\mu\upsilon}\eta^{\mu\upsilon}$ 。上述方程在线性近似下描述了引力波的传播与辐射。该方程需要补充规范变换 (4) $\delta h_{\mu\nu} = -\partial_\mu\xi_\nu - \partial_\nu\xi_\mu$ ，并要求施加规范固定条件，例如采用德唐德尔 (也叫福克-德唐德尔) 条件:

$$\partial_\mu h_\upsilon^\mu - \frac{1}{2}\partial_\upsilon h = 0. \tag{57}$$

Using condition (57) in Eq. (56), the last is cast in the form (see, e.g., [21])

将条件 (57) 代入式 (56) 后，方程可改写为如下形式 (参见例如文献 [21]):

$$\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}. \tag{58}$$

For the plane wave propagating along an arbitrary axis, the components of the metric perturbation (8), which are gauge invariant and can be called physical, are transverse ones, $\bar{h}^{\perp\perp}_{\mu\nu}$. Thus, the gravitational wave in GR is a propagation of the spin-2 state.

对于沿任意轴传播的平面波，度规微扰 (8) 中规范不变、可称为物理分量的部分是横分量，即 $\bar{h}^{\perp\perp}_{\mu\nu}$ 。因此广义相对论中的引力波是自旋 2 态的传播。

The emission of the gravitational wave in the linear regime corresponds to the solution in the standard form of the retarded potential,

线性区域下引力波的辐射对应推迟势标准形式的解，

$$h_{\mu\nu}(\mathbf{x}, t) = \frac{4}{M_P^2} \int d^3x' \frac{S_{\mu\nu}(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|}. \tag{59}$$

The factor $1/M_P^2 = G$ in this expression shows that the emission of the gravitational waves is suppressed by the square of the Planck mass. And after the wave travels a very long distance, there is a similar Planck suppression at the moment of its detection, which explains the difficulty of detecting the gravitational wave. The remarkable detection by LIGO is explained by the incredible quantity of energy emitted in the merger of two black holes or other extremely compact and massive objects, such as neutron stars.

该表达式中的因子 $1/M_P^2 = G$ 表明，引力波辐射被普朗克质量的平方压低。引力波传播极长距离后，探测过程也会存在类似的普朗克压低效应，这解释了引力波探测的难度。LIGO 之所以能够完成这一非凡的探测，是因为两个黑洞并合，或是中子星这类其他极端致密大质量天体并合时，会辐射出极其巨大的能量。

Quantization and the Notion of Graviton

量子化与引力子概念

At quantum level, the physical degrees of freedom corresponding to the state of a free gravitational field on a flat background correspond to the degrees of freedom of the linear gravitational wave described above. The corresponding particle with zero mass and spin-2 is called a graviton.

在量子层面，平直背景下自由引力场状态对应的物理自由度，与上述描述的线性引力波自由度一致。对应的零质量自旋为 2 的粒子称为引力子。

To derive the spin-2 part of the propagator of the gravitational perturbation $h_{\mu\nu}$, we can use the tensor part of the propagator (see, e.g., Eq. (87) below) and setting $\Phi = 0$. For the sake of simplicity, we omit the coefficient κ^2 and obtain the spin-2 part of the Euclidean propagator in the form

为推导引力微扰传播子的自旋 2 部分 $h_{\mu\nu}$ ，我们可以利用传播子的张量部分 (例如见下文式 (87))，并令 $\Phi = 0$ 。为简化起见，我们省略系数 κ^2 ，得到欧几里得传播子的自旋 2 部分形式如下

$$\langle h_{\mu\nu} h_{\alpha\beta} \rangle^{(2)} = G^{(2)}_{\mu\nu,\alpha\beta}(k) = \frac{P^{(2)}_{\mu\nu,\alpha\beta}(k)}{k^2}, \tag{60}$$

where the projector $P^{(2)}_{\mu\nu,\alpha\beta}(k)$ to the spin-2 states will be defined below. This equation describes the propagation of the massless degrees of freedom associated with the spin-2 states in GR, which is the graviton.

其中投影到自旋 2 态的投影算符 $P^{(2)}_{\mu\nu,\alpha\beta}(k)$ 将在后文定义。该方程描述了广义相对论中与自旋 2 态相关联的无质量自由度的传播，这种自由度就是引力子。

In other models of QG, the propagator can be more complicated because Φ (and also Ψ , because there may be a relevant dynamics in the scalar sector) are typically nonzero. For instance, including the fourth-derivative terms, there may be the following two changes:

在其他量子引力模型中，传播子可以更为复杂，因为 Φ (还有 Ψ ，因为标量 sector 可能存在相关动力学) 通常不为零。例如，引入四阶导数项后，通常会产生以下两种变化:

(i) Instead of the unique massless pole in the propagator (60), there may be additional massive poles.

(i) 传播子 (60) 原本只有唯一的无质量极点，新增后可能出现额外的质量极点。

(ii) The scalar components of the metric perturbation (8) gain a massive, gauge-independent sector in the propagator.

(ii) 度规微扰 (8) 的标量分量在传播子中获得一个有质量、规范无关的 sector。

Adding more derivatives, which means using polynomial or even nonlocal models, the modifications always concern the same two points. Namely, there will be (in the polynomial models) a growing number of poles in the spin-2 sector and the scalar sector. In contrast, some choices of nonlocal action may provide that there would not be any massive poles in the tree-level propagator, in both spin-2 and spin- 0 sectors.

引入更多导数，即使用多项式甚至非局域模型，修改始终围绕上述两点。也就是说，(多项式模型中) 自旋 2 sector 和标量 sector 的极点数量会不断增加。反之，某些非局域作用量的选择可以保证，树级传播子中的自旋 2 sector 和自旋 0 sector 都不存在任何质量极点。

It is worth noting that the count of degrees of freedom, based on the simple analysis of the gravitational propagator, was confirmed by the canonical quantization of the gravitational theory in the cases of quantum GR and fourth-derivative quantum gravity (see, e.g., [3]).

值得注意的是，在量子广义相对论和四阶导数量子引力的情形中，基于引力传播子简单分析得到的自由度计数结果，已经得到引力理论正则量子化的验证 (例如见 [3])。

Propagator of Metric and the Barnes-Rivers Projectors

度规传播子与 Barnes-Rivers 投影算符

At this point we note a common aspect of all mentioned models of QG, namely, the ones based on the actions (22),(30),(39). Since the propagator of $h_{\mu\upsilon}$ in flat background is defined by the quadratic in curvature (Riemann, Ricci, or scalar R) terms in the action, the form factors depending on $\Box$ do not influence the tensor structure of the bilinear form.

至此我们注意到，上述所有基于作用量 (22)、(30)、(39) 的量子引力模型都存在一个共通点。由于平坦背景下 $h_{\mu\upsilon}$ 的传播子由作用量中曲率二次项 (黎曼曲率、里奇曲率或标量 R) 定义，依赖 $\Box$ 的形状因子不会影响双线性型的张量结构。

The propagator G of the quantum metric, in any QG model, obeys the equation

在任意量子引力模型中，量子度规的传播子 G 满足方程

$$H^{\mu\upsilon,\alpha\beta}(x)\,G_{\alpha\beta,\rho\sigma}(x,y) = \delta^4(x-y)\,\delta^{\mu\upsilon}{}_{,\rho\sigma}, \tag{61}$$

where

其中

$$H^{\mu\upsilon,\alpha\beta}(x)\,\delta^4(x-y) = \frac{1}{2\sqrt{-g}}\frac{\delta^2 S}{\delta g_{\mu\upsilon}(x)\,\delta g_{\alpha\beta}(y)} \tag{62}$$

is the bilinear in quantum field form of the classical action S of a model of quantum gravity. The action in (62) should include the gauge-fixing term, as otherwise we meet a degeneracy. For the sake of definiteness, we start from the gauge-invariant action and denote the corresponding degenerate bilinear form coming from the initial action, as $H_{(0)}^{\mu\upsilon,\alpha\beta}$, while the nondegenerate version, after the Faddeev-Popov procedure, will be denoted as $H^{\mu\upsilon,\alpha\beta}$.

是量子引力模型经典作用量 S 的量子场双线性形式。(62) 式的作用量必须包含规范固定项，否则会出现简并。为明确起见，我们从规范不变作用量出发，将初始作用量给出的对应简并双线性形式记为 $H_{(0)}^{\mu\upsilon,\alpha\beta}$ ，经过法捷耶夫-波波夫 procedure 后得到的非简并版本记为 $H^{\mu\upsilon,\alpha\beta}$ 。

In this part of the chapter, all spacetime indices are raised and lowered with the flat background metric. Then, independently of the model and the choice of the gauge fixing, the operator $H^{\mu\upsilon,\alpha\beta}(x)$ has the following tensor structure:

本章此部分中，所有时空指标均通过平坦背景度量升降。因此，无论模型和规范固定选择如何，算符 $H^{\mu\upsilon,\alpha\beta}(x)$ 都具有如下张量结构:

$$H_{\mu\upsilon,\alpha\beta}(x;g) = a_1\delta_{\mu\upsilon,\alpha\beta}\Box + a_2\eta_{\mu\upsilon}\eta_{\alpha\beta}\Box + a_3\left(\eta_{\mu\upsilon}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\upsilon\right)$$

$$+a_4\left(\eta_{\mu\alpha}\partial_\beta\partial_\upsilon + \eta_{\upsilon\alpha}\partial_\beta\partial_\mu + \eta_{\mu\beta}\partial_\alpha\partial_\upsilon + \eta_{\upsilon\beta}\partial_\alpha\partial_\mu\right) - a_5\partial_\alpha\partial_\beta\partial_\mu\partial_\upsilon, \tag{63}$$

where $a_k = a_k(-\Box)$ are five model-dependent functions of the d’Alembert operator. In the higher-derivative cases, all of these functions are proportional to the linear combinations of $\Phi(\Box)$ and $\Psi(\Box)$ in Eq.

(39). In particular, for the fourth-derivative model (30), $a_{1,2,3,4}$ are linear functions of $\Box$ and a_5 = const. In the case of quantum GR, there are the constant functions $a_{1,\dots,4}$ and $a_5 = 0$. In what follows, we consider the general analysis of the propagator, which is valid for all types of models.

其中 $a_k = a_k(-\Box)$ 是五个依赖达朗贝尔算符的模型相关函数。在高阶导数情形下，所有这些函数都正比于 (39) 式中 $\Phi(\Box)$ 和 $\Psi(\Box)$ 的线性组合。特别地，对于四阶导数模型 (30)，$a_{1,2,3,4}$ 是 $\Box$ 和常数 $a_5 =$ 的线性函数。对于量子广义相对论，这些是常数函数 $a_{1,\dots,4}$ 和 $a_5 = 0$ 。下文我们将对传播子进行通用分析，结论适用于所有类型的模型。

Making a Fourier transform, we can rewrite the bilinear form in the momentum representation,

做傅里叶变换后，我们可以将双线性形式改写为动量空间形式，

$$H_{\mu\nu,\alpha\beta}(k;\eta) = -\left[a_1(k^2)\,\delta_{\mu\nu,\alpha\beta}k^2 + a_2(k^2)\,\eta_{\mu\nu}\eta_{\alpha\beta}k^2\right.$$

$$+a_3(k^2)\left(\eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu\right) \tag{64}$$

$$\left.+a_4(k^2)\left(\eta_{\mu\alpha}k_\beta k_\nu + \eta_{\nu\alpha}k_\beta k_\mu + \eta_{\mu\beta}k_\alpha k_\nu + \eta_{\nu\beta}k_\alpha k_\mu\right) + a_5(k^2)\,k_\alpha k_\beta k_\mu k_\nu\right],$$

where $k^2 = k_\mu k^\mu$ is the square of the four-dimensional momentum and $\delta_{\mu\nu,\alpha\beta}$ is similar to (48), but this time it is constructed from the flat metric $\eta_{\mu\nu}$.

其中 $k^2 = k_\mu k^\mu$ 是四维动量的平方，$\delta_{\mu\nu,\alpha\beta}$ 与 (48) 式形式类似，只不过此处由平坦度规 $\eta_{\mu\nu}$ 构造。

It is useful to present (64) in a slightly different form, providing more generality by using the n -dimensional versions of the formulas,

将 (64) 式改写为略有不同的形式会更便于处理，利用 n 维公式可以得到更强的通用性，

$$\hat{H} = s_1\hat{T}_1 + s_2\hat{T}_2 + s_3\hat{T}_3 + s_4\hat{T}_4 + s_5\hat{T}_5, \tag{65}$$

where $\hat{T}_n = T^{(n)}_{\mu\nu,\alpha\beta}$ and

其中 $\hat{T}_n = T^{(n)}_{\mu\nu,\alpha\beta}$ 且

$$\hat{T}_1 = \delta_{\mu\nu,\alpha\beta},\ \hat{T}_2 = \eta_{\mu\nu}\eta_{\alpha\beta},\ \hat{T}_3 = \frac{1}{k^2}\left(\eta_{\mu\nu}k_\alpha k_\beta + \eta_{\alpha\beta}k_\mu k_\nu\right), \tag{66}$$

$$\hat{T}_4 = \frac{1}{4k^2}\left(\eta_{\mu\alpha}k_\beta k_\nu + \eta_{\nu\alpha}k_\beta k_\mu + \eta_{\mu\beta}k_\alpha k_\nu + \eta_{\nu\beta}k_\alpha k_\mu\right),\ \hat{T}_5 = \frac{1}{k^4}k_\alpha k_\beta k_\mu k_\nu.$$

The coefficients depend on momentum, $s_l = s_l(k^2)$, and these dependencies may be nontrivial, e.g., in the polynomial or nonlocal models. However, the tensor structure of the expressions (65) is the same for all QG models, i.e., for quantum GR, or for a higher-derivative polynomial, or nonlocal models (39).

系数依赖于动量 $s_l = s_l\left(k^2\right)$，且这种依赖可能是非平凡的，例如多项式模型或非局域模型就是如此。但 (65) 式表达式的张量结构对所有量子引力模型都一致，即对量子广义相对论、高阶导数多项式模型或非局域模型 (39) 都成立。

To invert the operator (64) and take care about its possible degeneracy, consider the operators called Barnes-Rivers projectors [22, 23]. The starting point is to formulate the projectors to the transverse and longitudinal subspaces of the vector space. In the momentum representation we have

为对算符 (64) 求逆并处理其可能的简并性，我们引入名为 Barnes-Rivers 投影算符的算子 [22,23]。其构造出发点是给出向量空间横子空间和纵子空间的投影算符。在动量表象中我们有:

$$\omega_{\mu\nu} = \frac{k_\mu k_\nu}{k^2}, \; \theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \tag{67}$$

with the standard properties

满足标准性质

$$\omega_{\mu\nu}\omega^{\nu}{}_{\lambda} = \omega_{\mu\lambda}, \; \theta_{\mu\nu}\theta^{\nu}{}_{\lambda} = \theta_{\mu\lambda}, \; \omega_{\mu\nu}\theta^{\nu}{}_{\lambda} = 0. \tag{68}$$

Then, the projectors to the spin-2, spin-1, and spin-0 states in the symmetric tensor space are written in the form

随后，对称张量空间中自旋 2、自旋 1 和自旋 0 态的投影算符可写为如下形式

$$\hat{P}^{(2)} = P^{(2)}_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}\right) - \frac{1}{n-1}\theta_{\mu\nu}\theta_{\alpha\beta},$$

$$\hat{P}^{(1)} = P^{(1)}_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\beta}\omega_{\mu\alpha}\right),$$

$$\hat{P}^{(0-s)} = P^{(0-s)}_{\mu\nu,\alpha\beta} = \frac{1}{n-1}\theta_{\mu\nu}\theta_{\alpha\beta}, \; \hat{P}^{(0-w)} = P^{(0-w)}_{\mu\nu,\alpha\beta} = \omega_{\mu\nu}\omega_{\alpha\beta}. \tag{69}$$

On top of these, to have the closed algebra of projectors in the scalar sector, one needs the additional transfer operators

除此之外，为得到标量区封闭的投影算符代数，还需要引入额外的转移算符

$$\hat{P}^{(ws)} = P^{(ws)}_{\mu\nu,\alpha\beta} = \frac{1}{\sqrt{n-1}}\theta_{\mu\nu}\omega_{\alpha\beta}, \; \hat{P}^{(sw)} = P^{(sw)}_{\mu\nu,\alpha\beta} = \frac{1}{\sqrt{n-1}}\omega_{\mu\nu}\theta_{\alpha\beta}. \tag{70}$$

The algebra for the vector and tensor projectors is simple,

矢量和张量投影算子的代数很简单,

$$\hat{P}^{(2)}\hat{P}^{(i)} = \hat{P}^{(2)}\delta_{i2} \;\; \text{and} \;\; \hat{P}^{(1)}\hat{P}^{(i)} = \hat{P}^{(1)}\delta_{i1}, \tag{71}$$

where $i = (2, 1, 0-w, 0-s, sw, ws)$. In the scalar sector, one has to construct the matrix projector operator

其中 $i = (2, 1, 0-w, 0-s, sw, ws)$ 。在标量部分，我们需要构造矩阵投影算子

$$\hat{P}_0 = \frac{1}{2} \begin{Vmatrix} P^{(0-s)} & P^{(sw)} \\ P^{(ws)} & P^{(0-w)} \end{Vmatrix} \tag{72}$$

satisfying the relation $\hat{P}_0^2 = \hat{P}_0$. To end this part, the last two terms, which represent the scalar sector of (8), can be written, in momentum representation, as

满足关系 $\hat{P}_0^2 = \hat{P}_0$ 。在本部分的最后，代表式 (8) 标量部分的最后两项可以在动量表象下写为

$$h_{\mu\upsilon}^{\text{scalar}} = \frac{1}{4} h\theta_{\mu\upsilon} + \left(\frac{1}{4}h - k^2\varepsilon\right)\omega_{\mu\upsilon}, \tag{73}$$

such that acting by each of the two projectors $P^{(0-s)}$ and $P^{(0-w)}$, one of these terms remains invariant and another vanish.

因此两个投影算子 $P^{(0-s)}$ 和 $P^{(0-w)}$ 分别作用时，其中一项保持不变，另一项为零。

Now we are in a position to find the propagator. Solving Eq. (61) requires the inversion of the expression

现在我们可以来推导传播子了。求解式 (61) 需要对如下表达式求逆

$$\hat{H} = b_2\hat{P}^{(2)} + b_1\hat{P}^{(1)} + b_{os}P^{(0-s)} + b_{ow}\hat{P}^{(0-w)} + b_{sw}\left[P^{(ws)} + \hat{P}^{(sw)}\right], \tag{74}$$

which means one has to find such an operator

也就是说我们需要找到这样一个算子

$$\hat{G} = c_2\hat{P}^{(2)} + c_1\hat{P}^{(1)} + c_{os}P^{(0-s)} + c_{ow}\hat{P}^{(0-w)} + c_{sw}\left[P^{(ws)} + \hat{P}^{(sw)}\right], \tag{75}$$

where the product with $\hat{B}$ is unity,

它与 $\hat{B}$ 的乘积为单位算子,

$$\hat{H}\hat{G} = \hat{1} = \hat{P}^{(2)} + \hat{P}^{(1)} + P^{(0-s)} + \hat{P}^{(0-w)}. \tag{76}$$

Using the aforementioned algebra of projectors, we get the solution to this problem,

利用上述投影算子的代数，我们得到该问题的解为

$$c_2 = \frac{1}{b_2},\ c_1 = \frac{1}{b_1},\ c_{os} = -\frac{b_{ow}}{\Delta},\ c_{ow} = -\frac{b_{os}}{\Delta},\ c_{sw} = \frac{b_{sw}}{\Delta}, \tag{77}$$

where $\Delta = b_{sw}^2 - b_{os}b_{ow}$.

其中 $\Delta = b_{sw}^2 - b_{os}b_{ow}$ 。

It is clear that the action of a generally covariant theory before adding the gauge-fixing term has either $b_1 = 0$ or $\Delta = 0$. From this perspective, the purpose of the Faddeev-Popov procedure is to remove this degeneracy.

显然，广义协变理论在添加规范固定项之前，要么是 $b_1 = 0$ 要么是 $\Delta = 0$ 。从这个角度来看，法捷耶夫-波波夫过程的目的就是消除这种简并。

It remains to present the projectors (69) and the transfer operators (70) as the linear combinations of the expressions (66), and v.v. The first set of formulas is

接下来需要将投影算子 (69) 和传递算子 (70) 表示为表达式 (66) 的线性组合，反之亦然。第一组公式为

$$\widehat{P}^{(2)} = \widehat{T}_1 - \frac{1}{n-1}\widehat{T}_2 + \frac{1}{n-1}\widehat{T}_3 - 2\widehat{T}_4 + \frac{n-2}{n-1}\widehat{T}_5,\ \widehat{P}^{(1)} = 2\left(\widehat{T}_4 - \widehat{T}_5\right),$$

$$\widehat{P}^{(0-w)} = \widehat{T}_5,\ \widehat{P}^{(0-s)} = \frac{1}{n-1}\left(\widehat{T}_2 - \widehat{T}_3 + \widehat{T}_5\right),$$

$$P^{(ws)} + P^{(sw)} = \frac{1}{\sqrt{n-1}}\left(\widehat{T}_3 - 2\widehat{T}_5\right). \tag{78}$$

Finally, by inverting these relations, one can express the matrices in (66) as

最后，对这些关系求逆，即可将式 (66) 中的矩阵表示为

$$\widehat{T}_1 = \widehat{P}^{(2)} + \widehat{P}^{(1)} + P^{(0-s)} + \widehat{P}^{(0-w)},$$

$$\widehat{T}_2 = (n-1)\,P^{(0-s)} + \sqrt{n-1}\left[P^{(ws)} + \widehat{P}^{(sw)}\right] + \widehat{P}^{(0-w)},$$

$$\widehat{T}_3 = \sqrt{n-1}\left[P^{(ws)} + \widehat{P}^{(sw)}\right] + 2\widehat{P}^{(0-w)},$$

$$\widehat{T}_4 = \frac{1}{2}\widehat{P}^{(1)} + \widehat{P}^{(0-w)},\ \widehat{T}_5 = \widehat{P}^{(0-w)}. \tag{79}$$

Correspondingly, the transformations between the coefficients of (65) and (74) are given by the inverse relations to the ones of ”basic vectors,” i.e.,

相应地，式 (65) 和 (74) 系数之间的变换由“基向量”关系的逆关系给出，即

$$b_2 = s_1 + s_2 + s_3 + s_4,\ b_1 = (n-1)\,s_3 + \sqrt{n-1}s_5 + s_4,$$

$$b_{0s} = \sqrt{n-1}s_5 + 2s_4,\ b_{0w} = \frac{1}{2}s_2 + s_4,\ b_{sw} = s_4 \tag{80}$$

and

和

$$s_1 = b_2 - \frac{1}{n-1}b_1 + \frac{1}{n-1}b_{0s} - 2b_{0w} + \frac{n-2}{n-1}b_{sw},\ s_2 = \left(b_{0w} - b_{sw}\right),$$

$$s_3 = \frac{1}{n-1}\left(b_1 - b_{0s} + b_{sw}\right),\ s_4 = b_{sw},\ s_5 = \frac{1}{\sqrt{n-1}}\left(b_{0s} - 2b_{sw}\right). \tag{81}$$

The solution to Eq. (61) consists of casting the bilinear form in the standard form (66), using the relations between $\widehat{T}_l$ and projectors (79), inverting the result using (77) and, finally, using the inverse relations (78). In principle, this procedure works for the bilinear form of the total action (63) for an arbitrary model of quantum gravity if the chosen gauge-fixing term makes the bilinear form nondegenerate. Usually, the Faddeev-Popov procedure is sufficient in this respect, but if the original theory had an extra symmetry (e.g., the conformal one), one needs to apply an additional gauge fixing, e.g., setting $h = h^\mu{}_\mu = 0$ [24].

式 (61) 的求解步骤为: 利用 $\widehat{T}_l$ 和投影算子 (79) 之间的关系将双线性型改写为标准形式 (66)，再通过 (77) 对结果求逆，最后利用逆关系 (78)。原则上，只要所选的规范固定项能保证双线性型非退化，该流程适用于任意量子引力模型总作用量 (63) 的双线性型。通常来说，法捷耶夫-波波夫过程就足以满足要求，但如果原理论存在额外对称性 (例如共形对称性)，就需要额外进行规范固定，例如设定 $h = h^\mu{}_\mu = 0$ [24]。

To put the described procedure in practice, one has to use definitions (35), with $\Box^n \to \Phi$, and (40) and insert the form factors Φ, Ψ , and Ω into expansions (51), (52), (50), and (48). The bilinear form of the general gauge-fixing term (31) with the weight (43) can be easily calculated,

要实践上述流程，我们需要代入定义 (35)(其中 $\Box^n \to \Phi$) 和 (40)，再将形状因子 Φ, Ψ 和 Ω 代入展开式 (51)、(52)、(50) 和 (48)。带权重 (43) 的一般规范固定项 (31) 的双线性型很容易计算，

$$H^{(GF)}_{\mu\nu,\alpha\beta}(k;\eta) = \widehat{H}_{GF} = W\left(-k^2\right)k^4\left\{\beta^2(\gamma-1)\widehat{T}_2 + \beta(1-\gamma)\widehat{T}_3 - \frac{1}{4}\widehat{T}_4 + \gamma\widehat{T}_5\right\}. \tag{82}$$

Summing up all the terms, including the bilinear form of the original action $H_{(0)\mu\nu,\alpha\beta}(k;\eta)$, the contribution of (40) and of the gauge-fixing term, we arrive at the expression (65)

对所有项求和，包括原作用量 $H_{(0)\mu\nu,\alpha\beta}(k;\eta)$ 的双线性形式、式 (40) 的贡献以及规范固定项的贡献，我们得到表达式 (65)

$$\widehat{H} = \widehat{H}_{(0)} + \widehat{H}_{GB} + \widehat{H}_{GF}, \tag{83}$$

with the coefficients

其中系数为

$$s_1 = \frac{1}{2}\Phi k^4 + \frac{1}{2\kappa^2}k^2 + \frac{\Lambda}{\kappa^2},$$

$$s_2 = \left(\Psi - \frac{1}{6}\Phi\right)k^4 - \frac{1}{2\kappa^2}k^2 - \frac{\Lambda}{2\kappa^2} + \beta^2(\gamma-1)Wk^4,$$

$$s_3 = \left(\frac{1}{6}\Phi - \Psi\right)k^4 + \frac{1}{2\kappa^2}k^2 + \beta(1-\gamma)Wk^4,$$

$$s_4 = -\Phi k^4 - \frac{1}{\kappa^2}k^2 - Wk^4$$

$$s_5 = \left(\frac{1}{3}\Phi + \Psi\right)k^4 + \gamma Wk^4. \tag{84}$$

The remarkable detail is that there is no $\Omega\left(-k^2\right)$ in these expressions. This is certainly an expected result because this function comes from the "generalized" topological invariant (40), but we observe that this feature holds for any $\Omega\left(-k^2\right)$, not only for a constant, when this term in the action is really topological.

值得注意的是，这些表达式中不存在 $\Omega\left(-k^2\right)$。这当然是一个预期结果，因为该函数来自“广义”拓扑不变量 (40)，但我们观察到这一性质对任意 $\Omega\left(-k^2\right)$ 都成立，并非只有当作用量中的该项真正拓扑时才成立，也并非只有常数 $\Omega\left(-k^2\right)$ 才满足。

Another expected characteristic of expressions (84) is that all the coefficients except s_1 depend on the gauge-fixing parameters. Since (83) has three linearly independent coefficients $\frac{1}{4\alpha}, \frac{\beta\gamma}{\alpha}$, and $\frac{1-\gamma}{\alpha}$, by using the choice of the three parameters α, β, and γ, one can eliminate terms with the coefficients $a_{3,4,5}$ in the bilinear form (83). As a result, in any QG model, one can provide the minimal form of the total bilinear operator,

式 (84) 的另一个符合预期的特点是，除 s_1 外，所有系数都依赖于规范固定参数。由于式 (83) 有三个线性无关系数 $\frac{1}{4\alpha}, \frac{\beta\gamma}{\alpha}$ 和 $\frac{1-\gamma}{\alpha}$，通过选择三个参数 α, β 和 γ，我们可以消去双线性形式 (83) 中系数为 $a_{3,4,5}$ 的项。因此，在任意量子引力模型中，我们都可以得到总双线性算符的最小形式，

$$H^{\text{total}\,,\,\text{minimal}}_{\mu\nu,\alpha\beta}(k;\eta) = -\left[a_1\left(k^2\right)\delta_{\mu\nu,\alpha\beta} + a_2'\left(k^2\right)\eta_{\mu\nu}\eta_{\alpha\beta}\right]k^2, \tag{85}$$

where a_2' differs from a_2 in Eq. (64) because of the contribution of the gauge-fixing term. This property of the bilinear form holds also for an arbitrary background metric and is important for the one- or higher-loop calculations in QG.

其中 a_2' 与式 (64) 中的 a_2 不同，这是规范固定项贡献导致的。该双线性形式的性质对任意背景度规都成立，对量子引力的单圈或多圈计算都十分重要。

The main advantage of the minimal form (85) compared to the general one is that the minimal operators are directly suited for the use of the heat-kernel technique. Indeed, it is possible to work with the nonminimal operators in quantum gravity, e.g., using the generalized Schwinger-DeWitt technique [25], but it is always simpler to work with the minimal bilinear forms.

与一般形式相比，最小形式 (85) 的主要优势在于最小算符可直接用于热核技术。诚然，量子引力中也可以处理非最小算符，例如使用广义施温格-德维特技术 [25]，但处理最小双线性形式始终更简单。

In some QG models, the choice of parameters is more restricted. For example, in quantum GR, we meet only two gauge-fixing parameters, i.e., α and β , but there are only four nonminimal terms because of $a_5 = 0$. As a result, one can choose α and β to achieve the minimal form (85). One technical observation is that, if the initial action includes, simultaneously, a higher derivative (Φ - and Ψ -terms) and the Einstein-Hilbert action, one can provide minimality only in the higher-order terms (typically, in all of them, greatly simplifying calculations in the superrenormalizable models), but not in the second-derivative sector of the operator.

在部分量子引力模型中，参数的选择受到更多限制。例如，在广义相对论量子化中，我们只有两个规范固定参数，即 α 和 β ，且由于 $a_5 = 0$ ，仅存在四个非最小项。因此，我们可以选择 α 和 β 得到最小形式 (85)。一个技术结论是: 如果初始作用量同时包含高阶导数 (Φ 项和 Ψ 项) 和爱因斯坦-希尔伯特作用量，那么仅能在高阶项中实现最小性 (通常对所有高阶项都能实现，这会大大简化超可重整化模型的计算)，而无法在算符的二阶导数部分实现最小性。

Let us now analyze the situation from another perspective. Expressions (84) remain the same in any dimension n . Thus, we can use (80) to transform the operator $\hat{H}$ into the form (74). The result of this transformation is

现在我们从另一个角度分析该问题。表达式 (84) 在任意维数 n 下都保持不变。因此我们可以利用式 (80) 将算符 $\hat{H}$ 变换为形式 (74)，变换结果为

$$
\begin{aligned}
b_2 &= \frac{1}{2}\Phi k^4 + \frac{1}{2\kappa^2}k^2 + \frac{\Lambda}{\kappa^2},\\
b_1 &= -\frac{1}{2}Wk^4 + \frac{\Lambda}{\kappa^2},\\
b_{0s} &= -\frac{n-4}{6}\Phi k^4 + (n-1)\left[\Psi + \beta^2(\gamma-1)W\right]k^4 - \frac{n-2}{2\kappa^2}k^2 - \frac{(n-3)\Lambda}{2\kappa^2},\\
b_{0w} &= (1-\beta)^2(\gamma-1)Wk^4 + \frac{k^2}{\kappa^2} + \frac{\Lambda}{2\kappa^2},\\
b_{sw} &= \sqrt{n-1}\left[\beta(\beta-1)(\gamma-1)k^4W - \frac{\Lambda}{2\kappa^2}\right].
\end{aligned}
\tag{86}
$$

Once again, only the spin-2 coefficient is gauge-fixing independent.

再次说明，只有自旋 2 系数与规范固定无关。

The inversion formulas (77) are trivial in the spin-2 and spin-1 sectors. In the tensor sector, we get, for the $\Lambda = 0$ case, independent on the dimension n and on the gauge fixing,

反演公式 (77) 在自旋 2 和自旋 1 扇区是平凡的。对于张量扇区，在 $\Lambda = 0$ 情形下，我们得到与维数 n 、规范固定都无关的结果:

$$
c_2(\Lambda = 0) = \frac{2\kappa^2}{k^2(1+\kappa^2k^2\Phi)}. \tag{87}
$$

The vector part depends on the gauge fixing and has no direct physical interpretation. Thus, we concentrate on the results in the scalar sector. The reader can easily obtain the complete formulas, but since these formulas are cumbersome, we shall present only the qualitative results and the most interesting expression. In the cases of $\Lambda \neq 0$ or $n \neq 4$, all the scalar coefficients are gauge-fixing dependent. However, in case $\Lambda = 0$ and $n = 4$, one important coefficient is invariant (I am grateful to Dr. Leslaw Rachwalfor indicating to me this feature.),

> 矢量部分依赖于规范固定，没有直接物理解释。因此我们聚焦标量扇区的结果。读者可以很容易推导出完整公式，但由于这些公式十分繁琐，我们仅给出定性结果和最值得关注的表达式。在 $\Lambda \neq 0$ 或 $n \neq 4$ 情形下，所有标量系数都依赖于规范固定。然而在 $\Lambda = 0$ 和 $n = 4$ 情形下，存在一个不变的重要系数 (感谢 Leslaw Rachwal 博士向我指出这一性质),

$$c_{os}\left(\Lambda = 0, n = 4\right) = -\frac{\kappa^2}{k^2\left(1 - 3\kappa^2\Psi k^2\right)}. \tag{88}$$

It is easy to see from these formulas that the propagator (87) of the spin-2 mode depends only on the function Φ , while the spin-0 propagator (88) depends only on the function Ψ . Let us note that this output was anticipated already at the level of bilinear expansions of the classical action. Furthermore, both spin-2 and spin-0 propagators are gauge-fixing independent and contribute to the tree-level S -matrix of gravitational perturbations. It is interesting that these features, which are looking quite special, hold independent on the form of the form factors $\Phi\left(\Box\right)$ and $\Psi\left(\Box\right)$ in the action (39). The same concerns the irrelevance of the third form factor Ω for the propagator. Of course, this property is not valid for the vertices, if $\Omega\left(x\right)$ is not a constant function.

> 从这些公式中不难看出，自旋 2 模式的传播子 (87) 仅依赖于函数 Φ ，而自旋 0 传播子 (88) 仅依赖于函数 Ψ 。我们注意到，这一结果早在经典作用量的双线性展开阶段就已经得到了预测。此外，自旋 2 和自旋 0 传播子均与规范固定无关，会对引力微扰的树级 S 矩阵产生贡献。有趣的是，这些看似十分特殊的性质，与作用量 (39) 中形状因子 $\Phi\left(\Box\right)$ 和 $\Psi\left(\Box\right)$ 的具体形式无关。第三个形状因子 Ω 不影响传播子也是同理。当然，如果 $\Omega\left(x\right)$ 不是常数函数，该性质并不适用于顶点。

A peculiar situation occurs in the usual second-derivative gravity. Even if one starts from the pure GR, the result is the same as if setting $\Psi \to 0$ in (88), i.e.,(The factor of κ^2 in this formula appears because we used the expansion (9) with a flat background $g_{\mu\nu} = \eta_{\mu\nu}$. If using the expansion (24), there is no such coefficient.)

> 在常规二阶导数引力中会出现一种特殊情况。即使从纯广义相对论出发，得到的结果也和在 (88) 中令 $\Psi \to 0$ 的结果一致，即:(该公式中出现 κ^2 因子是因为我们使用了带平直背景 $g_{\mu\nu} = \eta_{\mu\nu}$ 的展开式 (9)。如果使用展开式 (24)，则不存在这个系数。)

$$c_{os}\left(GR \text{ with } \Lambda = 0\right) = -\frac{\kappa^2}{k^2}. \tag{89}$$

This formula is in apparent contradiction with what we saw in the analysis of the gravitational waves in GR, where the unique sort of the propagating degrees of freedom are the tensor (transverse and traceless) modes. The explanation of this apparent discrepancy is that the smooth $\Psi \to 0$ limit in (89) corresponds to the theory after the Faddeev-Popov procedure, which extends the space of the propagating modes and make the whole propagator nondegenerate. On the contrary, the result for the gravitational waves is based on another procedure, i.e., removing all degrees of freedom by using gauge transformation and its remnant (see, e.g., [21]).

该公式显然与我们在广义相对论引力波分析中得到的结论矛盾，在广义相对论中，唯一的传播自由度是张量 (横向无迹) 模式。解释这一明显矛盾的原因是:(89) 中的光滑 $\Psi \to 0$ 极限对应法捷耶夫-波波夫 procedure 之后的理论，该理论拓展了传播模式的空间，使整个传播子非退化。相反，引力波的相关结论是基于另一套 procedure 得到的，即利用规范变换及其剩余规范对称性移除所有非物理自由度 (参见例如文献 [21])。

From the QG perspective, the smooth limit (89) of the general expression (88) for the scalar sector is important, as it provides a universal IR limit of the propagator (for $n = 4$ only!) in any QG model (39) at the tree level, i.e., in both relevant sectors of the propagator.

从量子引力的角度来看，标量区一般表达式 (88) 的光滑极限 (89) 十分重要，因为它给出了任意量子引力模型 (39) 中传播子在树级的普适红外极限 (仅针对 $n = 4$!)，对传播子的两个相关区均成立。

Gauge-Invariant Renormalization in Quantum Gravity

量子引力中的规范不变重整化

In the next chapter of this section there is a detailed proof of the two main statements concerning the gauge-invariant renormalization in quantum gravity [26] (see also pioneer work [17] and [27], and more recently [28,29]).

本节的下一章节针对量子引力中规范不变重整化的两个核心命题给出详细证明 [26](另可参见开创性工作 [17] 与 [27]，以及近年研究 [28,29])。

Both theorems were already mentioned in the Introduction. However, since these two statements are relevant for the rest of this chapter, let us formulate them here in more detail:

两个定理已在引言中提及，但由于这两个命题对本章后续内容至关重要，我们在此给出更详细的表述:

1. The renormalization preserves the diffeomorphism invariance (general covariance) of the model of QG in four spacetime dimensions (i.e., in $n = 4$), if the initial classical theory possesses this symmetry. This means, one can remove the divergences in all loop orders by adding covariant counterterms. This statement applies literally only to the divergences that take place in the framework of the background field method, which is especially designed to avoid non-covariant counterterms. Let us stress that the background field method is not necessary for the gauge-invariant renormalization in QG. However, using the non-covariant parametrization of the metric, such as (24), one has to go through a relatively complicated procedure or additional renormalization transformations, as described in [17]. The final output is always the same in the sense of the same essential covariant counterterms. For this reason, in what follows we shall assume (24) when evaluating the power counting in different models of QG. However, we shall switch to the more general parametrization with the general background metric (9) when making the practical calculations.

1. 若初始经典理论拥有微分同胚不变性 (广义协变性)，那么在四维时空 (即 $n = 4$) 的量子引力模型中，重整化会保留该对称性。这意味着我们可以通过添加协变抵消项消除所有圈阶的发散。该表述严格来说仅适用于背景场方法框架下出现的发散，而背景场方法正是为避免非协变抵消项设计的。需要强调的是，背景场方法并非量子引力中规范不变重整化的必需方法。但若使用度规的非协变参数化 (例如式 (24))，就需要遵循 [17] 中描述的相对复杂的流程或额外的重整化变换。就核心协变抵消项而言，最终结果始终是一致的。因此，后续我们评估不同量子引力模型的幂次计数时将采用 (24) 的设定，而在实际计算中则会切换为更通用的、带有一般背景度规的参数化 (9)。

2. The dependence on the choice of the gauge condition (31), (e.g., on the parameters β, γ , and the function $W(\Box)$ in the weight operator (43) is proportional to the effective equations of motion. The same concerns the dependence of the parametrization of the quantum metric. In particular, both ambiguities vanish on the classical mass shell for the one-loop divergences of the effective action. In what follows (see more details in [7]) we shall demonstrate how this feature works in the practical calculations in QG.

2. 有效作用量对规范条件 (31) 选择的依赖 (例如对参数 β, γ 、权重算符 (43) 中函数 $W(\Box)$ 的依赖) 正比于有效运动方程。量子度规参数化选择的依赖也满足同样的关系。具体来说，对于有效作用量的单圈发散，两类不确定性在经典质壳上都会消失。后续我们将结合实例 (更多细节参见 [7]) 说明这一性质如何在量子引力的实际计算中发挥作用。

On top of that, there is the third statement, representing the general feature of QFT and valid, in particular, for QG. The counterterms required to remove UV divergences, in all loop orders, are local functionals of the fields. The mathematically rigid proof of this statement (usually called Weinberg's theorem [30]) is complicated and can be found, e.g., in [31]. In what follows, we shall apply these three statements to describe the renormalization in QG.

除此之外，还有第三个命题，它是量子场论的一般性特征，对量子引力同样成立: 消除所有圈阶紫外发散所需的抵消项都是场的局部泛函。该命题 (通常称为温伯格定理 [30]) 的数学严格证明较为复杂，例如可在 [31] 中找到。下文我们将应用这三个命题描述量子引力中的重整化。

Power Counting and Classification of Quantum Gravity Models

量子引力模型的幂次计数与分类

To estimate the power counting for the Feynman diagrams in QG is somehow simpler than in the quantum theories of other fields. The reason is that the metric is a dimensionless field. For this reason, the dimensions of the counterterm that emerge for diagram G , with L loops, is defined only by the number of derivatives of the background metric or, in case of a non-covariant parametrization such as (24), by the number of derivatives of $h_{\mu\nu}$. The power counting of a diagram is essentially equivalent to the count of dimensions and, therefore, it does not depend on the number of external lines of $h_{\mu\nu}$. Let us use the last version of the expansion, but with $\kappa \to 1$ for generality, as this enables us to include the higher-derivative theories into consideration.

估算量子引力中费曼图的幂次计数在一定程度上比其他场的量子理论更简单，原因在于度规是一个无量纲场。因此，图 G (含 L 圈) 涌现出的抵消项的量纲仅由背景度规的导数个数决定；若采用非协变参数化 (如式 (24))，则由 $h_{\mu\nu}$ 的导数个数决定。一个图的幂次计数本质上等价于量纲计数，因此它不依赖于 $h_{\mu\nu}$ 外线路的数量。我们采用展开的最新形式，但为了一般性引入 $\varkappa \to 1$，这样我们就能将高导数理论纳入考虑范围。

The superficial degree of divergences will be denoted $\omega(G)$ (sometimes it is also called the index of divergence) and $d(G)$ is the number of partial derivatives of the external lines of the field $h_{\mu\nu}$ in the diagram. Taking into account the powers of momenta in all elements of the diagrams (see, e.g., [7] for a detailed general treatment), the general expression is

我们将表观发散度记为 $\omega(G)$ (有时它也被称为发散指标)，其中 $d(G)$ 是图中场 $h_{\mu\nu}$ 外线路的偏导数个数。考虑图所有元素中的动量幂次 (例如，详细的通用处理可参见文献 [7])，得到的通用表达式为

$$\omega(G) + d(G) = \sum_{l_{\text{int}}} (4 - r_l) - 4V + 4 + \sum_V K_V, \tag{90}$$

where the first sum is over all I internal lines of the diagram, r_l is the inverse power of momentum in the propagator of an internal line, and V is the number of vertices. The last sum is taken over all the vertices, where K_V is the power of momenta,(or number of derivatives, in the coordinate representation) of all the lines coming to the given vertex.

其中第一项求和遍历图中所有 I 内线路，r_l 是内线路传播子中动量的逆幂次，V 是顶点个数。最后一项求和遍历所有顶点，K_V 是进入该顶点的所有线的动量幂次 (坐标表示中即导数个数)。

It is easy to see that formula (90) is insufficient to evaluate the renormalizability of the given QFT or QG model. In addition to this formula, there is the simple topological relation

不难看出，公式 (90) 不足以评估给定量子场论或量子引力模型的可重整化性。除该公式外，还有一个简单的拓扑关系

$$L = I - V + 1 \tag{91}$$

valid for all the relevant diagrams.

该关系对所有相关图都成立。

Before going on to consider concrete models of QG, let us make the following observation. The diagrams in quantum gravity, which we intend to analyze, have external lines of the field $h_{\mu\nu}$ only, but there are internal lines of both $h_{\mu\nu}$ and the Faddeev-Popov ghosts. However, with the modified definitions of the ghost actions (33), the values of r_l are the same for both kinds of quantum fields. For example, in quantum GR, in both cases we have $r_l = 2$, in fourth-derivative gravity in both cases $r_l = 4$, in the polynomial models $r_l = 2N + 4$. Finally, in the nonlocal QG models (39), both r_l and K_V are infinite for both metric and ghosts. Then the use of the combination of (90) and (91) is not possible. However, we shall see how to deal with this special case using the topological relation (91) alone. For a while, we assume that in all models of interest, r_l are identical for the quantum metric and the ghosts.

在开始讨论具体的量子引力模型之前，我们先给出下述观察结论: 我们打算分析的量子引力中的图，仅存在场 $h_{\mu\nu}$ 的外线路，但同时存在 $h_{\mu\nu}$ 和法捷耶夫-波波夫鬼的内线路。不过，在修改后的鬼作用量定义 (33) 下，两类量子场的 r_I 取值是相同的。例如，在广义相对论量子理论中，两种情况下都有 $r_I = 2$；在四导数引力中，两种情况下都有 $r_I = 4$；在多项式模型中为 $r_I = 2N + 4$。最后，在非局域量子引力模型 (39) 中，度规和鬼的 r_I 和 K_V 都是无穷大，因此无法结合使用 (90) 和 (91)。不过我们会看到，对于这种特殊情况，仅用拓扑关系 (91) 即可处理。暂时我们假设，在所有我们关心的模型中，量子度规和鬼的 r_I 是相同的。

Power Counting in Quantum Gravity Based on GR

基于广义相对论的量子引力幂次计数

As the first step, consider power counting in quantum GR, where $r_I = 2$ for all internal lines. The vertices coming from the Einstein-Hilbert term have $K_{EH} = 2$. If we include the cosmological constant term, there are also vertices $K_\Lambda = 0$. However, looking only for strongest divergences, at first we consider only the diagrams with $K_V = 2$ vertices. Then (90), together with the topological relation (91), yields

第一步，我们来考察量子广义相对论的幂次计数，其中所有内线满足 $r_I = 2$。来自爱因斯坦-希尔伯特项的顶点具有 $K_{EH} = 2$。如果我们纳入宇宙学常数项，还会存在顶点 $K_\Lambda = 0$。不过，若仅考虑最强发散，我们首先只讨论含 $K_V = 2$ 顶点的图。随后式 (90) 结合拓扑关系 (91) 可得

$$\omega(G) + d(G) = 2I - 4V + 4 + 2V = 2I - 2V + 4 = 2 + 2L. \qquad (92)$$

The last result clearly shows that the QG based on GR is non-renormalizable. The one-loop $L = 1$ and the logarithmic divergences with $\omega(G) = 0$ have $d(G) = 4$. Taking into account the diffeomorphism invariance, this indicates towards the counterterms repeating the covariant structures included in the fourth-derivative action (30). Indeed, at the one-loop order, there are counterterms of the Einstein-Hilbert form $\sim \int \sqrt{-g}R$ with $d(G) = 2$ but with quadratic divergences only, since $\omega(G) = 2$.

上述结果清楚表明，基于广义相对论的量子引力是不可重整化的。单圈 $L = 1$ 和带 $\omega(G) = 0$ 的对数发散具有 $d(G) = 4$。考虑微分同胚不变性后，这说明抵消项会重复四阶导数作用量 (30) 中包含的协变结构。实际上，在单圈阶确实存在爱因斯坦-希尔伯特形式的抵消项 $\sim \int \sqrt{-g}R$，对应 $d(G) = 2$，但仅存在二次发散，因为 $\omega(G) = 2$。

The logarithmic divergences of this type are also possible, but only if we introduce the cosmological constant term. If there is one vertex with $K_\Lambda = 0$, the diagram produces the logarithmic divergence with two derivatives. Assuming the covariance, this means an Einstein-Hilbert-type counterterm. With two such vertices, we meet a logarithmic divergence without derivatives, i.e., with $d(G) = 0$. In one of the next sections, we confirm these conclusions by direct calculations and also analyze the gauge-fixing and parametrization dependence of the one-loop counterterms.

这类对数发散也可能存在，但仅当我们引入宇宙学常数项时才会出现。如果存在一个含 $K_\Lambda = 0$ 的顶点，该图会产生带二阶导数的对数发散。在协变假设下，这对应一个爱因斯坦-希尔伯特型的抵消项。如果存在两个这样的顶点，我们会得到不带导数的对数发散，即对应 $d(G) = 0$ 。我们会在后续某一节中通过直接计算验证这些结论，同时分析单圈抵消项对规范固定和参数化的依赖。

One can rewrite the one-loop divergences using the relations

我们可以利用下述关系改写单圈发散:

$$C^2 = E_4 + 2W\left(\text{where } W = R_{\mu\upsilon}^2 - \frac{1}{3}R^2\right),\ E_4,\ R^2,\ \Box R. \tag{93}$$

We know that E_4 and $\Box R$ are surface terms, which do not affect the dynamics of the theory, and the other two terms vanish on the classical equations of motion, when $R_{\mu\upsilon} = 0$. Thus, the one-loop S -matrix in the pure quantum GR (pure means without matter contents) is finite. In the presence of matter this feature does not hold [32,33], but let us concern only pure QG.

我们知道 E_4 和 $\Box R$ 都是表面项，不会影响理论的动力学；当 $R_{\mu\upsilon} = 0$ 时，另外两项在经典运动方程上为零。因此，纯量子广义相对论 (纯指不包含物质内容) 中的单圈 S 矩阵是有限的。该性质在存在物质时不成立 [32,33]，因此我们仅讨论纯量子引力。

In the two-loop order $L = 2$. According to Eq. (92), the logarithmic divergences without K_Λ vertices have dimension six. A complete list of the corresponding terms has been elaborated in the works on the conformal anomaly in six spacetime dimensions [34]. This list includes

在两圈阶，$L = 2$ 。根据式 (92)，不含 K_Λ 顶点的对数发散量纲为六。六维时空共形反常的相关研究已经给出了对应项的完整列表 [34]，该列表包含

$$\sum\nolimits_1 = R_{\mu\nu}R^{\mu\alpha}R^{\nu}_{\alpha}\ \sum\nolimits_2 = \left(\nabla_\lambda R_{\mu\nu\alpha\beta}\right)^2\ \sum\nolimits_3 = R_{\mu\alpha\nu\beta}\nabla^\mu\nabla^\nu R^{\alpha\beta}$$

$$\sum\nolimits_4 = R_{\mu\nu}R^{\mu\lambda\alpha\beta}R^{\nu}{}_{\lambda\alpha\beta}\ \sum\nolimits_5 = R^{\mu\nu}{}_{\alpha\beta}R^{\alpha\beta}{}_{\lambda\tau}R^{\lambda\tau}{}_{\mu\nu}\ \sum\nolimits_6 = R^{\mu}{}_{\alpha}{}^{\nu}{}_{\beta}R^{\alpha}{}_{\lambda}{}^{\beta}{}_{\tau}R^{\lambda}{}_{\mu}{}^{\tau}{}_{\nu}$$

$$\sum\nolimits_7 = \left(\nabla_\lambda R_{\mu\upsilon}\right)^2\ \sum\nolimits_8 = R_{\mu\upsilon}\Box R^{\mu\upsilon}\ \sum\nolimits_9 = \left(\nabla_\mu R\right)^2$$

$$\sum\nolimits_{10} = R\Box R\ \sum\nolimits_{11} = \left(\nabla_\alpha R_{\mu\upsilon}\right)\nabla^\mu R^{\upsilon\alpha}\ \sum\nolimits_{12} = R^{\mu\upsilon}\nabla_\mu\nabla_\upsilon R$$

$$\sum\nolimits_{13} = R_{\mu\upsilon}R_{\alpha\beta}R^{\mu\alpha\upsilon\beta} \tag{94}$$

as well as the set of surface terms,

以及一组表面项,

$$\Xi_1 = \Box^2 R\ \Xi_2 = \Box R^2_{\mu\nu\alpha\beta}\ \Xi_3 = \Box R^2_{\mu\nu}\ \Xi_4 = \Box R^2$$

$$\Xi_5 = \nabla_\mu \nabla_\nu \left(R^\mu{}_{\lambda\alpha\beta} R^{\nu\lambda\alpha\beta} \right) \; \Xi_6 = \nabla_\mu \nabla_\nu \left(R_{\alpha\beta} R^{\mu\alpha\nu\beta} \right)$$

$$\Xi_7 = \nabla_\mu \nabla_\nu \left(R^\mu_\alpha R^{\nu\alpha} \right) \; \Xi_8 = \nabla_\mu \nabla_\nu \left(R R^{\mu\nu} \right), \tag{95}$$

satisfying the identity

满足恒等式

$$\Xi_2 - 4\Xi_3 + \Xi_4 - 4\Xi_5 + 8\Xi_6 + 8\Xi_7 - 4\Xi_8 = 0. \tag{96}$$

All these structures can show up in the two-loop divergences, but only two of these terms, namely, $\sum_5$ and $\sum_6$, are critically important because they do not vanish on shell. The two-loop calculation was done in [35] and confirmed in [36] by using another calculational approach. The results confirmed the nonzero coefficient of $\sum_5$. The conclusion is that there are no miracles and the theory of QG based on GR is non-renormalizable.

所有这些结构都可能出现在两圈发散中，但其中仅有两项，即 $\sum_5$ 和 $\sum_6$ 至关重要，因为它们在在壳条件下不为零。两圈计算最早由文献 [35] 完成，文献 [36] 使用另一种计算方法得到了验证结果，确认了 $\sum_5$ 的非零系数。结论是并没有奇迹发生，基于广义相对论的量子引力理论确实不可重整化。

Within the standard perturbative approach, the non-renormalizability means the theory has no predictive power. With every new order of the loop expansion, there are new types of local covariant divergences, with the growing number of derivatives of the metric. And every time when a new type of counterterm is introduced, it is necessary to fix the renormalization condition. Each of these conditions requires making a measurement and using its result to fix the value of the corresponding parameter. In quantum GR, this sequence of operations is formally infinite. Thus, before making a single prediction, it is necessary to use an infinite amount of experimental data.

在标准微扰论框架下，不可重整化意味着理论不具备预言能力。随着圈展开阶数的升高，会不断出现新型的定域协变发散，度规导数的数量也不断增加。每次引入新型抵消项后，都需要重新固定重整化条件。每一个条件都需要通过实验测量，利用测量结果确定对应参数的取值。在量子广义相对论中，这一操作序列形式上是无穷的。因此，在做出任何一个预言之前，我们都需要用到无穷多的实验数据。

What are the possible ways out of this situation? The main options are as follows:

解决这一困境可能有哪些方向？主要的可选方案如下：

1. Change the standard perturbative approach to something else. The reader can consult other sections of our handbook, to see how the problem is solved in the framework of non-perturbative approaches, superstring theory, etc. Let us say that there are many interesting options, but their consistency and the relation to the QG program are not completely clear, in all cases.

1. 将标准微扰方法替换为其他方案。读者可以参考本手册的其他章节，了解该问题在非微扰方法、超弦理论等框架下的解决方案。可以说目前存在许多值得探索的方向，但在所有情况下，它们的自洽性以及和量子引力研究纲领的关联都尚未完全阐明。

2. Restrict the area of application of QG to the low-energy domain. The reader can read about this possibility in the section about effective QG. The main problem with this approach is that the QG is initially supposed to be a concept describing extreme high-energy physics, with the typical energy scale of the Planck order of magnitude. It is certainly important to know what remains from the QG effects in the IR, but this does not reduce the importance of formulating QG that would be applicable at high energies.

2. 将量子引力的适用范围限制在低能区。读者可以在有效量子引力相关章节了解这一思路。该方法的核心问题在于，量子引力原本就应当是描述极端高能物理的概念，其典型能标为普朗克量级。了解量子引力效应在红外区的残留固然重要，但这并不会降低构建可适用于高能区的量子引力理论的重要性。

3. Change the theory, i.e., start from the model different from GR, to construct QG. This option is the mainstream direction in perturbative QG. It is important that the problems we meet in this way, such as the problem of nonphysical ghosts coming from higher-derivative terms, persist in many nontraditional approaches which we mentioned in the first point (see, e.g., the discussion in [37]).

3. 修改理论，即从不同于广义相对论的模型出发构建量子引力。这是微力量子引力领域的主流方向。需要注意的是，这种思路下遇到的问题，例如来自高阶导数项的非物理鬼问题，在我们第一点中提到的诸多非传统方法中依然存在 (例如参见文献 [37] 中的讨论)。

Power Counting in Fourth-Derivative Gravity Models

四导数引力模型中的幂次计数

The next example is the power counting in the fourth-derivative quantum gravity (30). We assume the Faddeev-Popov procedure with the second-order weight operator (32) and the modified action of ghosts (33). In this case, for all modes of the gravitational perturbation $h_{\mu\nu}$ and ghosts, we have $r_l = 4$, while the vertices K_V include K_{4d}, K_{EH} , and K_Λ .

下一个例子是四导数量子引力 (30) 中的幂次计数。我们假设采用带有二阶权重算符 (32) 的法捷耶夫-波波夫步骤，以及修正的鬼作用量 (33)。在这种情况下，对于引力扰动 $h_{\mu\nu}$ 和鬼的所有模式，我们有 $r_l = 4$ ，而顶点 K_V 包含 K_{4d}, K_{EH} 和 K_Λ 。

Let us denote n_{4d} the number of vertices with fourth power of momenta, n_{EH} the one with two, and n_Λ - with zero power of momenta. Then

我们记 n_{4d} 为带四次动量幂次的顶点数，n_{EH} 为带二次动量幂次的顶点数，n_Λ 为带零次动量幂次的顶点数。由此可得

$$n_{4d} + n_{EH} + n_\Lambda = V \text{ and } n_{4d}K_{4d} + n_{EH}K_{EH} + n_\Lambda K_\Lambda = \sum_V K_V. \tag{97}$$

The general expression (90), together with the topological relation (91), gives the following result:

通式 (90) 结合拓扑关系 (91)，给出如下结果:

$$\omega(G) + d(G) = 4 - 2n_{EH} - 4n_{\Lambda}. \qquad (98)$$

As a starting point, consider the diagrams with the strongest divergences, where all of the vertices are of the K_{4d} type, i.e., $V = n_{4d}$. In this case,(98) means that the logarithmic divergences have $d(G) = 4$. Taking into account locality and covariance arguments, the possible counterterms are of the C^2, R^2, E_4 , and $\Box R$ types. This means, in all loop orders the divergences have the same form as the fourth-derivative terms in the classical action (30). Then, for $n_{4d} = V - 1$ and $n_{EH} = 1$, we obtain $d(G) = 2$, corresponding to the counterterm linear in R . Finally, for $n_{EH} = 2$ and $V - 2 = n_{4d}$, or for $n_{\Lambda} = 1$ and $V - 1 = n_{4d}$, there is a counterterm with $d(G) = 0$, which is the cosmological constant. Thus, the theory under consideration is multiplicatively renormalizable. In the next sections, we shall see that this does not mean that the theory is completely consistent, as there is a massive nonphysical ghost in the spectrum and the subsequent problems with quantum unitarity and even with the stability of classical solutions.

作为出发点，考虑发散最强的图，这类图的所有顶点都是 K_{4d} 型，即 $V = n_{4d}$ 。在这种情况下，(98) 表明对数发散对应 $d(G) = 4$ 。结合局域性和协变性的论证，可能的抵消项属于 C^2, R^2, E_4 型和 $\Box R$ 型。这意味着，在所有圈阶，发散的形式都和经典作用量 (30) 中的四导数项形式相同。那么，对于 $n_{4d} = V - 1$ 和 $n_{EH} = 1$ ，我们得到 $d(G) = 2$ ，对应关于 R 线性的抵消项。最后，对于 $n_{EH} = 2$ 和 $V - 2 = n_{4d}$ ，或是 $n_{\Lambda} = 1$ 和 $V - 1 = n_{4d}$ ，存在带 $d(G) = 0$ 的抵消项，这就是宇宙学常数。因此，所讨论的理论是可乘可重整化的。在后文中我们会看到，这并不意味着该理论完全自洽，因为它的谱中存在一个有质量非物理鬼，还会进一步引发量子幺正性甚至经典解稳定性的问题。

For the particular case of the general model (30) without dimensional parameters, the classical action has a global conformal symmetry under the transformation $g_{\mu\nu} \to g_{\mu\nu}e^{2\lambda}$, with $\lambda =$ const. The power counting (98) can be perfectly well applied in this case, yielding $\omega(G) + d(G) = 4$. This means that the theory is also multiplicatively renormalizable. The disadvantage of this model of gravity is that there is no automatic Einstein limit in the low-energy domain. Let us remember that such a limit is one of the main conditions of consistency of any model which generalizes or modifies GR, so this situation should be seen as a problem of the model.

对于不含量纲参数的一般模型 (30) 这种特殊情况，经典作用量在变换 $g_{\mu\nu} \to g_{\mu\nu}e^{2\lambda}$ 下具有整体共形对称性，其中 $\lambda =$ 为常数。幂次计数公式 (98) 可以很好地应用于此情形，得到 $\omega(G) + d(G) = 4$ 。这说明该理论也是可乘可重整化的。这种引力模型的缺点在于，它在低能区不会自动得出爱因斯坦极限。我们要记住，对于任何推广或修改广义相对论的模型，这类极限都是其自洽性的核心条件之一，因此这种情况应被视为该模型的一个问题。

We note, by passing, that one can start from such a theory of globally conformal theory, coupled to a massless scalar field. At the quantum level, the loop corrections to the scalar potential may produce such effective potential that the global conformal symmetry is dynamically broken, producing GR in the low-energy limit. This idea resurrected several times in the literature (see, e.g., [38-40]) in different QG frameworks and, in general, looks attractive and promising. Unfortunately, the real deal is that, in the IR, one has to break the global conformal symmetry. And then, in the broken phase, we come back to the massive ghost problems and to the related issue of instabilities, as it will be discussed in the subsequent section "Massive Ghosts in Higher-Derivative Models."

顺便一提，我们完全可以从这类整体共形引力理论出发，将它与一个无质量标量场耦合。在量子层面，对标量势的圈修正可能会产生这样的有效势: 整体共形对称性发生动力学破缺，最终在低能极限得到广义相对论。这个想法在不同量子引力框架下的文献中被反复提出 (例如参见 [38-40])，整体来看颇具吸引力、很有前景。但不幸的是，实际情况是，我们必须在红外区破缺整体共形对称性。而后在破缺相，我们会再次遇到有质量鬼问题以及相关的不稳定性问题，这些内容我们会在后文“高阶导数模型中的有质量鬼”一节中讨论。

Another particular case, which is instructive to consider, is the $R + R^2$ -gravity, which is model (30) without the C^2 -term. As we saw in the previous sections, in this model the traceless component of the metric $\bar{h}_{\mu\nu}$ has $1/k^2$ propagator, while the scalar mode has a propagator behaving as $1/k^4$ in the UV. Furthermore, there are vertices K_{4d} connecting all these modes. It is easy to check that the power counting in this model is dramatically different from that in the general fourth-derivative model. The theory is non-renormalizable and the power counting is even much worse than in quantum GR.

另一个值得研究的特殊情况是 $R+R^2$ 引力，即不含 C^2 项的模型 (30)。正如我们在前几节中看到的，在该模型中，度规 $\bar{h}_{\mu\nu}$ 的无迹分量具有 $1/k^2$ 传播子，而标量模式的传播子在紫外区的行为为 $1/k^4$ 。此外，存在连接所有这些模式的顶点 K_{4d} 。不难验证，该模型的幂次计数与一般四导数引力模型存在极大差异。该理论是不可重整化的，其幂次计数的情况甚至比广义相对论量子理论更差。

The next example is the model (30) without the R^2 term. This particular model may be interesting since the fourth-derivative part of the action possesses local conformal symmetry. This symmetry is softly broken by the Einstein-Hilbert and cosmological terms. The expression ”soft breaking” means that the symmetry does not hold in the terms with dimensional parameters. Can it be that the softly broken conformal symmetry ”saves” the power counting in this model? The answer is certainly negative. The propagator of the traceless mode of the metric, $\bar{h}_{\mu\nu}$, in this case, has the UV behavior $1/k^4$ and that of the scalar mode the different UV behavior, $\propto 1/k^2$, due to the presence of the R -term. At the same time, there are K_{4d} vertices that link all of the modes, and hence, the power counting is qualitatively the same as in the previous $R+R^2$ case. The theory is not renormalizable.

下一个例子是不含 R^2 项的模型 (30)。这个特殊模型可能很有意义，因为其作用量的四导数部分具有局域共形对称性。该对称性被爱因斯坦-希尔伯特项和宇宙学项软破缺。“软破缺”指该对称性在含维度参数的项中不成立。被软破缺的共形对称性是否能“改善”该模型的幂次计数？答案当然是否定的。在这个模型中，度规的无迹模式 $\bar{h}_{\mu\nu}$ 的传播子在紫外区行为为 $1/k^4$ ，而由于 R 项的存在，标量模式的紫外区行为不同，为 $\propto 1/k^2$ 。同时，存在连接所有模式的顶点 K_{4d} ，因此幂次计数在定性上与前一个 $R+R^2$ 的情况相同，该理论仍是不可重整化的。

The situation is much more complicated if the original theory has the pure Weyl-squared term in the Lagrangian, i.e., possesses local conformal symmetry. The theoretical proof of the renormalizability in conformal theories exists only for the quantum theory in curved space (semiclassical gravity) [41] and, in the literature, there is no proof for conformal QG. On the other hand, at the one-loop order this theory is renormalizable, as was shown by direct calculations [24,42] and confirmed in [43]. However, it is expected that this model is not renormalizable at higher loops because of the conformal anomaly. But, in the situation when this expectation is not supported by direct higher-loop calculations or the analysis similar to [41], the question should be regarded as open.

如果原理论拉格朗日中仅包含纯外尔平方项，即本身具有局域共形对称性，那么情况会复杂得多。目前共形理论可重整化性的理论证明仅存在于弯曲空间量子理论 (半经典引力) 中 [41]，现有文献中尚未给出共形量子引力的相关证明。另一方面，正如直接计算 [24,42] 所展示、并经 [43] 验证的，该理论在单圈阶是可重整化的。但学界普遍认为，由于共形反常，该模型在多圈阶不可重整化。不过，目前这一观点尚未得到直接多圈计算或 [41] 这类分析的支持，因此该问题仍应视为开放问题。

Power Counting in the Polynomial Theory

多项式理论中的幂次计数

Consider power counting in the polynomial model (34). As before, we assume that the Faddeev-Popov quantization is done with the weight operator (37) and the correspondingly modified ghost term (33). In the general case, both coefficients of the Highest-derivative terms $\omega_{N,R}$ and $\omega_{N,C}$ are nonzero. Then the propagators of both metric perturbations $h_{\mu\nu}$ and ghosts have the UV behavior $\propto k^{-4-2N}$, i.e., we have $r_l \equiv 4+2N$. For the vertices, the generalization of Eq. (97) is

我们来研究多项式模型 (34) 中的幂次计数。和之前一样，我们假设采用法捷耶夫-波波夫量子化，使用权算子 (37) 以及相应修正的鬼项 (33)。一般情况下，最高导数项的两个系数 $\omega_{N,R}$ 和 $\omega_{N,C}$ 均非零。此时度规涨落 $h_{\mu\nu}$ 和鬼场的传播子都具有紫外行为 $\propto k^{-4-2N}$ ，即我们得到 $r_l \equiv 4+2N$ 。对于顶点，式 (97) 的推广形式为

$$\sum_V K_V = n_{4+2N}K_{4+2N} + n_{2+2N}K_{2+2N} + n_{2N}K_{2N} + \ldots + n_{EH}K_{EH} + n_\Lambda K_\Lambda,$$

$$V = n_{4+2N} + n_{2+2N} + n_{2N} + \ldots + n_{EH} + n_\Lambda, \tag{99}$$

where $K_{4+2N} = 4+2N, K_{2+2N} = 2+2N, \ldots K_{EH} = 2, K_\Lambda = 0$, and n_{4+2N} , $n_{2+2N}, n_{2N}, \ldots, n_{EH}$, and n_Λ are the numbers of the respective vertices.

其中 $K_{4+2N} = 4+2N, K_{2+2N} = 2+2N, \ldots K_{EH} = 2, K_\Lambda = 0$ 、n_{4+2N} 、$n_{2+2N}, n_{2N}, \ldots, n_{EH}$ 和 n_Λ 分别是对应顶点的数量。

Consider the diagrams with the strongest divergences. This means $V = n_{4+2N}$, such that the other types of vertices are absent. Then the power counting becomes quite simple, because of $\sum_V K_V = V(4+2N)$. The expression (90) becomes

我们来研究发散最强的图，这对应 $V = n_{4+2N}$ ，即不存在其他类型的顶点。此时由于 $\sum_V K_V = V(4+2N)$ ，幂次计数变得十分简单，式 (90) 变为

$$\omega(G) + d(G) = (4-4-2N)I - 4V + 4 + V(4+2N)$$

$$= 4 + 2N(V-I) = 4 + 2N(1-L), \tag{100}$$

where we used the topological relation (91) in the form $V - I = 1 - L$. It is easy to see that the power counting in the four-derivative model (98) is a particular case, corresponding to $N = 0$. Thus, we assume $N \geq 1$.

其中我们利用了拓扑关系 (91)，形式为 $V - I = 1 - L$ 。不难看出，四导数模型 (98) 中的幂次计数是对应 $N = 0$ 的特殊情况，因此我们假设 $N \geq 1$ 。

According to (100), the sum $\omega(G) + d(G)$ decreases with a growing number of loops L . The strongest divergences occur for $L = 1$, when the aforementioned sum equals 4 and the logarithmic divergences correspond to the one-loop counterterms with, at most, four derivatives. Taking the covariance and locality, this means that the one-loop counterterms are of the C^2, R^2, E_4 , and $\Box R$ types. In other words, all the terms with six and more derivatives do not need to be renormalized at the one-loop order. But, if there is just one vertex with two less derivatives, i.e., $n_{2+2N} = 1$ and $n_{4+2N} = V - 1$, then we meet only the Einstein-Hilbert-type divergence. And finally, in the case $n_{2+2N} = 2$ or $n_{2N} = 1$, the unique divergence is that of the cosmological constant.

根据 (100)，求和式 $\omega(G)+d(G)$ 随圈数 L 增加而减小。最强发散出现在 $L = 1$ 处，此时上述求和式等于 4，对数发散对应最多含四阶导数的单圈 counter 项。考虑协变性和局域性，这说明单圈 counter 项属于 C^2, R^2, E_4 和 $\Box R$ 类型。换言之，所有六阶及以上导数项都不需要在单圈阶进行重整化。但如果仅存在一个少两阶导数的顶点，即 $n_{2+2N} = 1$ 和 $n_{4+2N} = V - 1$ ，那么我们只会得到爱因斯坦-希尔伯特型发散。最后，在 $n_{2+2N} = 2$ 或 $n_{2N} = 1$ 的情况下，唯一的发散是宇宙学常数的发散。

The features of the $L = 1$ approximation that we listed above do not depend on $N \geq 1$. Starting from $L \geq 2$, the structure of divergences starts to depend on the value of N . In particular, for $N \geq 3$, according to (100), the second- and higher-loop diagrams are all finite. This creates a situation when the one-loop beta functions are the exact ones. In the case of $N = 2$, there are two-loop divergences, but only of the cosmological constant type. Finally, for $N = 1$, there are two-loop divergences of the Einstein-Hilbert and the cosmological constant type and also three-loop divergences, but only of the cosmological constant type.

我们上面列出的 $L = 1$ 近似的性质不依赖于 $N \geq 1$ 。从 $L \geq 2$ 开始，发散结构开始依赖于 N 的值。特别地，对于 $N \geq 3$ ，根据式 (100)，二阶及更高圈的图都是有限的，此时单圈 β 函数就是精确 β 函数。对于 $N = 2$ ，存在仅属于宇宙学常数类型的两圈发散。最后，对于 $N = 1$ ，存在爱因斯坦-希尔伯特型和宇宙学常数型的两圈发散，以及仅属于宇宙学常数类型的三圈发散。

All in all, a theory with both $\omega_{N,R} \neq 0$ and $\omega_{N,C} \neq 0$ is super-renormalizable. In contrast, in the degenerate case, when only one of these coefficients is zero, the theory is non-renormalizable. The situation is essentially similar to what we discussed above for two similar four-derivative models of QG.

总而言之，同时含有 $\omega_{N,R} \neq 0$ 和 $\omega_{N,C} \neq 0$ 的理论是超可重整化的。与之相反，在退化情形下，即其中一个系数为零，理论是不可重整化的。该情况和我们之前讨论的两个相似的四导数量子引力模型基本一致。

Finally, the power counting in the QG models (39) with non-polynomial (typically exponential) functions Φ and Ψ cannot be performed on the basis of the formula (90), because both the number of derivatives in the vertices and the parameter r_l are infinite. However, if the condition (41) is satisfied, the evaluation can be done using only the topological relation [44]. Let us assume, for simplicity, that the functions are those from (42). Then, in Euclidean signature, each propagator brings the exponential of $-\alpha k^2$ and each vertex contributes with the exponential of $+\alpha k^2$. This means, if the number of vertices is different from the number of propagators, the last integral in the given diagram will be either strongly divergent or completely convergent. Taking this into account, from relation (91) we learn that the divergences are present only in the one-loop diagrams and that these divergences have fourth powers of momenta. Thus, the power counting in the theories of this class is the same as in the polynomial models with $N \geq 3$.

最后，对于带有非多项式 (通常为指数型) 函数 Φ 和 Ψ 的量子引力模型 (39)，无法基于公式 (90) 进行幂次数计数，因为顶点中的导数数量与参数 r_l 均为无穷大。但如果满足条件 (41)，则仅可利用拓扑关系完成该计算 [44]。为简化起见，我们假设这里的函数就是 (42) 给出的函数。那么在欧几里得号差下，每个传播子都会带来 $-\alpha k^2$ 的指数项，每个顶点贡献 $+\alpha k^2$ 的指数项。这意味着，如果顶点数与传播子数不相等，给定图的最终积分要么强发散要么完全收敛。据此，我们从关系 (91) 可知，发散仅存在于单圈图中，且这类发散对应动量的四次方。因此，这类理论的幂次数计数与带有 $N \geq 3$ 的多项式模型一致。

Massive Ghosts in Higher-Derivative Models

高阶导数模型中的大质量鬼

Previously we saw that quantum GR, based on the Einstein-Hilbert action, is a non-renormalizable theory. On the other hand, by adding the general fourth-derivative terms, we arrive at the model providing multiplicative renormalizability. Another strong argument for including fourth-derivative terms is that they are required for the renormalizability of the semiclassical theory, when gravity is an external field [2, 3, 5, 7] . And this is something one cannot disregard. The point is that the concepts of QFT from one side and of the curved space from another are well-established and, to a great extent, verified by experiments and observations. Thus, if QFT in curved space produces the four-derivative terms in the action, we have to admit that these terms are there. The problem concerns not exactly UV divergences and formal renormalizability. Together with the logarithmic divergences, there are always logarithmic nonlocal form factors. In the IR such a form factor behaves, effectively, as a constant [7, 53]. This means, if we do not include fourth derivatives into the action, they will come there anyway as legitimate corrections coming from quantum matter fields. Therefore, independent on which approach to QG we choose, it makes sense to include fourth-derivative terms into the gravitational action and check out the physical consequences of this inclusion.

此前我们已经看到，基于爱因斯坦-希尔伯特作用量的量子广义相对论是不可重整理论。另一方面，通过引入一般四阶导数项，我们可以得到具有可乘重整性的模型。支持引入四阶导数项的另一个有力论据是，当引力作为外场时，半经典理论的可重整化性要求这类项存在 [2, 3, 5, 7] ，这一点是我们无法忽视的。问题的关键在于，量子场论和弯曲空间各自都是已经确立的概念，并且在很大程度上得到了实验和观测的验证。因此，如果弯曲空间中的量子场论会在作用量中产生四阶导数项，我们就必须承认这类项是真实存在的。问题并不仅仅关乎紫外发散和形式上的可重整性。除了对数发散，始终还存在对数非定域形状因子。在红外区域，这种形状因子实际上表现为常数 [7, 53]。这意味着，即使我们一开始不把四阶导数项写入作用量，它们也会作为量子物质场带来的合理校正自然出现在作用量中。因此，无论我们选择哪种量子引力方案，在引力作用量中引入四阶导数项并检验其引入带来的物理效应都是合理的。

At the classical level and in the low-energy domain, the fourth derivative terms look irrelevant because the coefficients of these terms in the action (30) are just a number, while the coefficient of the R -term is $1/G$, that is, $M_P^2 \approx 10^{38}\mathrm{GeV}$. As a consequence, until the metric derivatives, in the momentum representation, do not have Planck-order frequencies, the fourth-derivative terms cannot compete with the Einstein-Hilbert term. This situation is usually called a Planck suppression. So, from the first sight the deal is perfect: including the fourth-derivative terms we get a renormalizable QG (and semiclassical too!), while all classical solutions remain the same as in GR and one can enjoy the well-verified gravitational theory.

在经典层面和低能区域，四阶导数项看起来无关紧要，因为这类项在作用量 (30) 中的系数只是一个无量纲数，而 R 项的系数为 $1/G$ ，即 $M_P^2 \approx 10^{38}\mathrm{GeV}$。因此，在动量表示中，只要度量导数不达到普朗克量级的频率，四阶导数项就无法与爱因斯坦-希尔伯特项相比。这种情况通常被称为普朗克压低。因此乍一看一切都很完美: 引入四阶导数项后我们得到了可重整的量子引力 (就连半经典理论也可重整！)，同时所有经典解都保持和广义相对论中一致，我们可以继续使用这套得到充分验证的引力理论。

Unfortunately, even if theory (30) is, formally, multiplicatively renormalizable, it does not make it consistent at either the quantum or even classical levels. As we shall see in brief, the spectrum of this model includes states that have negative kinetic energy. These states, or particles, are called massive nonphysical ghosts. The presence of ghosts violates the unitarity of the theory at the quantum level. Worse than that, in the presence of massive ghosts or, more generally, ghostlike states, classical solutions of the theory can be unstable with respect to the metric perturbations. Qualitatively, the same situation takes place not only in the fourth-derivative theory but in all polynomial models of quantum gravity.

遗憾的是，即使理论 (30) 形式上满足可乘重整性，这并不意味着它在量子层面甚至经典层面都是自洽的。我们很快就会看到，该模型的能谱中包含动能项为负的态，这些态 (粒子) 被称为大质量非物理鬼。鬼的存在会破坏理论在量子层面的幺正性。更糟糕的是，在存在大质量鬼，或者更一般地说存在类鬼态的情况下，该理论的经典解可能对度量摄动不稳定。从定性上看，不仅四阶导数理论，所有量子引力多项式模型都会出现这种情况。

The problem of ghosts is certainly the main obstacle for building a consistent QG theory. For this reason, we consider this problem here. We shall closely follow [7]. The interested reader can address this book for more details, or go directly to the original works, such as the reviews [45] on the Ostrogradsky instabilities, or the papers [46] and [47-49] for the approach we follow to explore and interpret the instabilities caused by massive ghosts.

鬼问题无疑是构建自洽量子引力理论的主要障碍。正因如此，我们在此专门讨论这个问题，我们的讨论将严格遵循文献 [7]。感兴趣的读者可以参考这本书获取更多细节，也可以直接查阅原始文献: 比如关于奥斯特罗格拉德斯基不稳定性的综述 [45]，或是我们用于探索和解释大质量鬼引发的不稳定性的相关文献 [46] 以及 [47-49]。

What Means a Massive Ghost

什么是质量鬼

Consider the propagator of the transverse and traceless part of the metric (87) or the one of the scalar, gauge-invariant mode (88). For simplicity, we can set the cosmological constant to be zero, and then the fourth-derivative action (30) becomes

考虑度规横向无迹部分的传播子 (87)，或标量规范不变模的传播子 (88)。为简化起见，我们可将宇宙常数设为零，此时四阶导数作用量 (30) 变为

$$S=-\int d^4x\sqrt{-g}\left\{\frac{1}{\lambda}R_{\mu\nu}^2+\frac{\omega-1}{3\lambda}R^2+\frac{1}{\kappa^2}R\right\}. \tag{101}$$

Formulas (87) and (88) change accordingly, with Φ and Ψ becoming constants.

公式 (87) 和 (88) 相应变化，Φ 和 Ψ 变为常数。

We can consider at once both scalar and tensor modes, because the formulas are similar. For the definiteness sake, consider the tensor mode $h=\bar{h}_{\mu\nu}^{\perp\perp}$, which is not affected by the R^2 term. Using the expansions (50) and (48), the action of this mode becomes

由于公式形式相似，我们可以同时讨论标量模和张量模。为明确起见，考虑不受 R^2 项影响的张量模 $h=\bar{h}_{\mu\nu}^{\perp\perp}$ 。利用展开式 (50) 和 (48)，该模的作用量变为

$$S^{(2)}_{\text{tensor}}=\int d^4x\left\{-\frac{1}{4\lambda}(\Box h)^2-\frac{1}{4\kappa^2}h\Box h\right\}=-\frac{1}{4\lambda}\int d^4x h\left(\Box+m_2^2\right)\Box h, \tag{102}$$

where $m_2^2=\lambda/\kappa^2$ is the mass of the mode that is called a tensor ghost, a massive tensor ghost or higher-derivative ghost. The reason for this exotic name is that the Euclidean propagator of the spin-2 mode in this theory can be cast in the form

其中 $m_2^2=\lambda/\kappa^2$ 是该模的质量，该模被称为张量鬼、质量张量鬼或高阶导数鬼。这个特殊名称的由来是，该理论中自旋 2 模的欧几里得传播子可以写为如下形式

$$G_2(k)\propto\frac{1}{m_2^2}\left(\frac{1}{k^2}-\frac{1}{k^2+m_2^2}\right)\hat{P}^{(2)}. \tag{103}$$

The negative sign of the second term indicates that the corresponding mode is not a usual particle. In fact, we have not one but two degrees of freedom of the tensor field. One of these degrees of freedom has positive kinetic energy and zero mass, and it corresponds to the first term in Eq. (103). The second degree of freedom has the mass m_2 and corresponds to the second term in (103). As we shall see in what follows, its kinetic energy is negative, and, for this reason, it is called a ghost.

第二项的负号表明对应模不是普通粒子。实际上张量场共有两个自由度，而非一个。其中一个自由度动能为正、质量为零，对应式 (103) 中的第一项。第二个自由度质量为 m_2 ，对应式 (103) 中的第二项。下文我们会说明，它的动能为负，因此被称为鬼。

The separation of the two degrees of freedom can be most simply explored by using an auxiliary field Φ (see, e.g.,[50], a more detailed discussion in [51], and more general formulations in [52]). Consider the Lagrangian density

两个自由度的分离可以通过引入辅助场 Φ 最简单地实现 (例如参见 [50]，更详细的讨论见 [51]，更一般的表述见 [52])。考虑拉格朗日密度

$$\mathcal{L}' = -\frac{m_2^2}{4\lambda} h\Box h + \lambda\phi^2 - \phi\Box h. \tag{104}$$

The Lagrange equation for ϕ can be solved as $\phi = \frac{1}{2\lambda}\Box h$. Substituting this expression back into (104), we arrive at Eq. (102), which shows the dynamical equivalence of the models (102) and (104).

ϕ 的拉格朗日方程可解为 $\phi = \frac{1}{2\lambda}\Box h$ 。将该表达式代回 (104)，我们得到式 (102)，这说明模型 (102) 和 (104) 动力学等价。

The two fields Ψ and h in (104) are not factorized. To improve on this issue, we change to the new variables θ and ψ ,

(104) 中的两个场 Ψ 和 h 没有分离。为解决这个问题，我们换用新变量 θ 和 ψ ,

$$h = \frac{\sqrt{2\lambda}}{m_2}\left(a_1\theta + a_2\psi\right),\ \phi = \frac{\sqrt{2\lambda}}{m_2} a_3\psi, \tag{105}$$

where the unknown coefficients $a_{1,2,3}$ should provide the separation of the modes and also the standard coefficients $\frac{1}{2}$ or $-\frac{1}{2}$ in the kinetic terms. A small algebra shows that the condition $a_2 + a_3 = 0$ is necessary to separate the variables and, also, the condition $a_1 = a_2 = 1$ is required to provide standard normalization of the kinetic terms. Then, in the new variables, the Lagrangian (104) becomes

其中未知系数 $a_{1,2,3}$ 需要满足模分离的要求，同时保证动能项具有标准系数 $\frac{1}{2}$ 或 $-\frac{1}{2}$ 。简单计算可知，分离变量要求条件 $a_2 + a_3 = 0$ 成立，而动能项的标准归一化要求条件 $a_1 = a_2 = 1$ 成立。此时，拉格朗日量 (104) 在新变量下变为

$$\mathcal{L}' = \frac{1}{2}\eta^{\mu\upsilon}\partial_\mu\theta\partial_\upsilon\theta - \frac{1}{2}\left(\eta^{\mu\upsilon}\partial_\mu\psi\partial_\upsilon\psi - m_2^2\psi^2\right). \tag{106}$$

We can conclude that the theory (30) has healthy tensor massless degrees of freedom θ and, on top of this, tensor massive degrees of freedom ψ with negative kinetic energy, called nonphysical massive ghost.

我们可以得出结论: 理论 (30) 除了具有良好定义的无质量张量自由度 θ 之外，还存在动能为负的质量张量自由度 ψ ，被称为非物理质量鬼。

Classification of Ghosts and Tachyons

鬼与快子的分类

Consider a basic classification of ghosts and tachyons following [7,53]. The general action of a free second-order field $h(x) = h(t, \mathbf{r})$ can be written as

我们沿用文献 [7,53] 给出鬼与快子的基础分类。自由二阶场 $h(x) = h(t, \mathbf{r})$ 的一般作用量可以写为

$$S(h) = \frac{s_1}{2} \int d^4x \{\eta^{\mu\upsilon}\partial_\mu h \partial_\upsilon h - s_2 m^2 h^2\}$$
$$= \frac{s_1}{2} \int d^4x \left\{\dot{h}^2 - (\nabla h)^2 - s_2 m^2 h^2\right\}. \tag{107}$$

Here s_1 and s_2 are sign factors ± 1 for different types of fields. In what follows, we consider all four combinations of these signs.

此处 s_1 和 s_2 是不同类型场对应的符号因子 ± 1 。下文中我们将讨论这两个符号全部四种组合的情况。

It is useful to perform the Fourier transform in the space variables,

对空间变量做傅里叶变换会更便于分析,

$$h(t, \mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k e^{i\mathbf{k}\cdot\mathbf{r}} h(t, \mathbf{k}). \tag{108}$$

In a free theory, one can consider the dynamics of each component $h \equiv h(t, \mathbf{k})$ separately. Such a dynamics is defined by the action

在自由理论中，可以单独分析每个分量 $h \equiv h(t, \mathbf{k})$ 的动力学。该动力学由如下作用量定义

$$S_{\mathbf{k}}(h) = \frac{s_1}{2} \int dt \{\dot{h}^2 - \mathbf{k}^2 h^2 - s_2 m^2 h^2\} = \frac{s_1}{2} \int dt \{\dot{h}^2 - m_k^2 h^2\}, \tag{109}$$

$$\text{where } \mathbf{k}^2 = \mathbf{k} \cdot \mathbf{k} \text{ and } m_k^2 = s_2 m^2 + \mathbf{k}^2 . \tag{110}$$

The properties of the field are defined by the signs of s_1 and s_2 . The possible options can be classified as follows:

场的性质由 s_1 和 s_2 的符号决定，可能的情况分类如下:

(i) A normal healthy field corresponds to $s_1 = s_2 = 1$. The kinetic energy of the field is positive and the equation of motion has the oscillatory form,

(i) 常规正常场对应 $s_1 = s_2 = 1$ 。该场动能为正，运动方程为振荡形式

$$\ddot{h} + m_k^2 h = 0, \tag{111}$$

with the usual periodic solutions.

解为常规周期解。

(ii) A tachyon has $s_1 = 1$ and $s_2 = -1$. The classical dynamics of tachyons is described in the literature, e.g., in [54,55], but, for our purposes, it is sufficient to give only a basic survey. For a relatively small momentum $k = |\mathbf{k}|$, there is $m_k^2 < 0$ in Eq. (110), and the equation of motion is

(ii) 快子满足 $s_1 = 1$ 且 $s_2 = -1$ 。已有文献如 [54,55] 描述过快子的经典动力学，本文仅需给出基础概述即可。当动量 $k = |\mathbf{k}|$ 相对较小时，式 (110) 中存在 $m_k^2 < 0$ ，运动方程为

$$\ddot{h} - \omega^2 h = 0, \ \omega^2 = |m_k^2|, \tag{112}$$

with exponential solutions

解为指数形式

$$h = h_1 e^{\omega t} + h_2 e^{-\omega t}. \tag{113}$$

However, if such a particle moves faster than light, the solution is of the normal oscillatory kind, indicating that such a motion is ”natural” for this kind of particle.

但若这类粒子的运动速度超过光速，解就是常规振荡形式，说明这种运动对这类粒子而言是「自然」的。

(iii) A massive ghost has $s_1 = -1$ and $s_2 = 1$. It is not a tachyon, because $m_k^2 \geq 0$. In this case, the kinetic energy of the field is negative, but the Lagrange equation (leaving aside its derivation from the least action principle) has a normal oscillatory equation (111).

(iii) A 质量鬼满足 $s_1 = -1$ 且 $s_2 = 1$ 。它不是快子，因为 $m_k^2 \geq 0$ 。在这种情况下，场的动能为负，但拉格朗日方程 (暂不讨论其从最小作用量原理的推导) 仍是常规的振荡方程 (111)。

(iv) A tachyonic ghost has $s_1 = s_2 = -1$. For relatively small $\mathbf{k}^2$, one meets $m_k^2 < 0$. The kinetic energy is negative and the mass is imaginary. So, along with the problems typical for the ghosts, the free wave solutions are exponential, as in (113). One can find a discussion of the implication of tachyonic ghosts in [53].

(iv) 快子鬼满足 $s_1 = s_2 = -1$ 。当 $\mathbf{k}^2$ 相对较小时，可得 $m_k^2 < 0$。其动能为负，质量为虚数。因此，除了鬼本身固有的问题外，自由波解也和 (113) 一样是指数形式。关于快子鬼的相关讨论可以参见文献 [53]。

Massive Ghosts in the Fourth-Order Model

四阶模型中的大质量鬼

Let us come back to the fourth-order gravity model (101). Consider the free tensor modes in the theory (102) and change to the momentum representation,

我们回到四阶引力模型 (101)。考虑该理论 (102) 中的自由张量模，并转换到动量表象，

$$\Box h = \ddot{h} - \Delta h \to \ddot{h} + \mathbf{k}^2 h. \tag{114}$$

Here $\mathbf{k}$ is the wave vector of an individual mode. It is important that, owing to the presence of both massless and massive modes, the standard massless dispersion relation between the frequency and the wave vector does not hold in this case. The Lagrange function of the wave with fixed $\mathbf{k}$ can be obtained from (102):

此处 $\mathbf{k}$ 是单个模式的波矢。需要注意，由于同时存在无质量模和大质量模，标准的频率与波矢间的无质量色散关系在此情形下不成立。固定 $\mathbf{k}$ 后，波的拉格朗日函数可由 (102) 得到:

$$L = -\frac{1}{4\lambda}\left(\ddot{h} + \mathbf{k}^2 h\right)^2 - \frac{1}{4\kappa^2} h\left(\ddot{h} + \mathbf{k}^2 h\right). \tag{115}$$

The Lagrange equation for $L = L(q, \dot{q}, \ddot{q})$ has the form

$L = L(q, \dot{q}, \ddot{q})$ 对应的拉格朗日方程形式为

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} + \frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot{q}} = 0 \tag{116}$$

and the energy can be easily obtained in the form

能量可以很容易地写成如下形式

$$E = \dot{q}\left(\frac{\partial L}{\partial \dot{q}} - \frac{d}{dt}\frac{\partial L}{\partial \ddot{q}}\right) + \ddot{q}\frac{\partial L}{\partial \ddot{q}} - L. \tag{117}$$

In our case (115), this formula gives the energy of the individual wave with momentum $\mathbf{k}$,

在我们的情况 (115) 中，该公式给出动量为 $\mathbf{k}$ 的单个波的能量,

$$E = \frac{1}{4\lambda}\left(2h^{(\mathrm{III})}\dot{h} - \ddot{h}^2\right) + \left(\frac{1}{4\kappa^2} + \frac{\mathbf{k}^2}{2\lambda}\right)\dot{h}^2 + \left(\frac{\mathbf{k}^2}{4\kappa^2} + \frac{\mathbf{k}^4}{4\lambda}\right)h^2. \tag{118}$$

This formula provides some information about the fourth-derivative theory. We can separate it into the following points:

该公式为四阶导数理论提供了若干信息，我们可以将其分为以下几点:

(i) In the limit $\lambda \to \infty$, the remaining expression for the energy is positively defined, as it should be for Einstein's gravity.

(i) 在 $\lambda \to \infty$ 极限下，能量的剩余表达式是正定的，符合爱因斯坦引力的要求。

(ii) The fourth time derivative terms are given by the first summand in (118). It is easy to see that this term is not positively defined. This sign indefiniteness should be expected, as a direct consequence of the presence of the massive nonphysical ghost.

(ii) 四阶时间导数项由 (118) 中的第一个求和项给出。不难看出，该项不是正定的。这种符号不确定性是存在大质量非物理鬼的直接结果，本就在预料之中。

(iii) In the model under discussion, the low-energy limit (IR) means

(iii) 在我们讨论的模型中，低能极限 (红外) 意味着

$$\ddot{h}^2 \ll \mathbf{k}^2\dot{h}^2 \quad \text{and} \quad \left|hh^{(\mathrm{III})}\right| \ll \mathbf{k}^2\dot{h}^2. \tag{119}$$

In this case, the first indefinite term (with fourth derivatives) in (118) is small, and the sign of the energy is defined by the second term, providing a relevant constraint on the action (30). The positivity of the theory in this limit does not depend on fourth time derivatives. However, the kinetic energy can be still unbounded from below for the negative coupling $\lambda < 0$. Owing to the violated dispersion relation between the wave vector $\mathbf{k}$ and the time derivatives, it is possible to have a large $\mathbf{k}^2$ with the conditions (119) satisfied. Thus, the sign of the coupling λ in the action (30) should be positive, as it was always assumed in the literature, e.g., in the classical works [17, 18] and [24].

这种情况下，(118) 中第一个不定项 (含四阶导数) 很小，能量的符号由第二项决定，该项给作用量 (30) 给出了一个相关约束。该极限下理论的能量正定性不依赖于四阶时间导数。然而，对于负耦合 $\lambda < 0$，动能仍可以没有下界。由于波矢 $\mathbf{k}$ 和时间导数之间的色散关系被破坏，在满足条件 (119) 的情况下仍可以存在大的 $\mathbf{k}^2$ 。因此，作用量 (30) 中耦合 λ 的符号应当为正，这和文献中一直以来的假设一致，例如经典工作 [17, 18] 和 [24] 中就是如此。

The equation for tensor perturbations can be derived from (116)(The more general equation describing the dynamics of tensor perturbations on the cosmological background will be discussed below; see Eq. (128).),

张量扰动方程可以从 (116) 导出 (描述宇宙学背景下张量扰动动力学的更一般方程会在后文讨论, 参见式 (128)),

$$h^{(\mathrm{IV})} + 2\mathbf{k}^2\ddot{h} + \mathbf{k}^4 h + \frac{\lambda}{16\pi\kappa^2}\left(\ddot{h} + \mathbf{k}^2 h\right) = 0. \tag{120}$$

One can introduce the new notation,

我们可以引入新的记号,

$$\frac{\lambda}{16\pi\kappa^2} = s_2 m^2 \tag{121}$$

where $s_2 = \operatorname{sign}\lambda$ and $m^2 > 0$. Then Eq. (120) becomes

其中 $s_2 = \operatorname{sign}\lambda$ 和 $m^2 > 0$ 。此时式 (120) 变为

$$\left(\frac{\partial^2}{\partial t^2} + \mathbf{k}^2\right)\left(\frac{\partial^2}{\partial t^2} + m_k^2\right)h = 0, \tag{122}$$

where m_k^2 was defined in (110). The solutions of the last equation can be different, depending on the sign of λ and, hence, that of s_2 . The general formula for the frequencies in $h \sim \exp\{\{\pm\omega t\}\}$ has the form

其中 m_k^2 已在 (110) 中定义。上述方程的解依 λ 的符号 (以及由此导出的 s_2 的符号) 而不同。$h \sim \exp\{\{\pm\omega t\}\}$ 中频率的通式为

$$\omega_{1,2} \approx \pm i\left(\mathbf{k}^2\right)^{1/2} \text{ and } \omega_{3,4} \approx \pm\left(-m_k^2\right)^{-1/2}. \tag{123}$$

For a positive λ , there are only imaginary $\omega' s$ and, hence, oscillator-type solutions. In contrast, for $\lambda < 0$, we have $s_2 = -1$ and the roots $\omega_{3,4}$ are real, since, in this case, $-m_k^2 > 0$ for sufficiently small $\mathbf{k}^2$. Indeed, the first couple of roots corresponds to the massless graviton and the second couple to the massive particle. According to the classification presented above, this particle is a ghost for $\lambda > 0$ and it is a tachyonic ghost for $\lambda < 0$.

当 λ 为正时，仅存在虚数 $\omega' s$ ，因此只有振子型解。相反，当为 $\lambda < 0$ 时，我们有 $s_2 = -1$ 且根 $\omega_{3,4}$ 为实数，因为在这种情况下，对于足够小的 $\mathbf{k}^2$ 有 $-m_k^2 > 0$ 。实际上，第一对根对应无质量引力子，第二对根对应大质量粒子。根据上述分类，当 $\lambda > 0$ 时该粒子是鬼，当 $\lambda < 0$ 时该粒子是快鬼。

Finally, we conclude that the model (30) has ghosts (and maybe tachyonic ghosts) owing to the presence of fourth derivatives.

最后，我们得出结论: 由于四阶导数的存在，模型 (30) 存在鬼 (可能还有快子鬼)。

Massive Ghosts in the Six and Higher-Order Models

六阶及更高阶模型中的大质量鬼

If the number of derivatives in the polynomial model is six or greater, the structure of the propagator is more complicated than in the fourth-order theory. First of all, the massive poles can be either real or complex. In the last case, they emerge in the complex conjugate pairs. There is an important theorem about the structure of the propagator in the case of real poles [20]. In this case, instead of (103), we meet

如果多项式模型中的导数阶数为六阶或更高，其传播子的结构会比四阶理论更复杂。首先，大质量极点可以是实极点，也可以是复极点。复极点会以复共轭对的形式出现。关于实极点情况下传播子的结构，存在一个重要定理 [20]。此时我们会得到如下形式，取代式 (103):

$$G_2(k) = \frac{A_0}{k^2} + \frac{A_1}{k^2 + m_1^2} + \frac{A_2}{k^2 + m_2^2} + \ldots + \frac{A_{N+1}}{k^2 + m_{N+1}^2}, \tag{124}$$

where the squares of the masses m_j^2 are real. Assuming that there is the following hierarchy of the masses:

其中质量平方 m_j^2 均为实数。假设质量满足如下层级关系:

$$0 < m_1^2 < m_2^2 < m_3^2 < \ldots < m_{N+1}^2, \tag{125}$$

one can prove that the signs of the coefficients A_j alternate, i.e., sign $\left[A_j\right]$ = - sign $\left[A_{j+1}\right]$. This feature means one cannot choose the theory in a such a way that all degrees of freedom instead of the heaviest one are normal particles and the mass of the ghost is infinitely large.

可以证明系数 A_j 的符号交替，即符号 $\left[A_j\right]$ = -符号 $\left[A_{j+1}\right]$。这一性质意味着，我们无法构造出这样的理论: 除最重自由度外，所有自由度均为正常粒子，且鬼的质量无穷大。

The complex poles are also possible, but their detailed discussion lies beyond the scope of this review. Interesting hints about the role of real and complex poles come from the analysis of the Newtonian limit and the bending of light in the polynomial models [37, 56, 57] .

复极点也可能存在，对其的详细讨论超出了本综述的范围。关于实极点和复极点的作用，可从多项式模型 [37, 56, 57] 的牛顿极限与光偏折分析中得到有趣的启示。

On the Quantum Consistency and Stability of Classical Solutions

经典解的量子一致性与稳定性

We have seen that the theory with fourth derivatives, typically, has a massive ghost or a tachyonic ghost, and this conclusion can be extended to the polynomial models with more than four derivatives. So, it is interesting to understand what the ghost means, from the physical viewpoint. Let us give just a brief qualitative description of the situation.

我们已经知道，四阶导数理论通常存在一个有质量鬼场或快子鬼场，这一结论可以推广到导数阶数高于四的多项式模型。因此，从物理角度理解鬼场的含义是很有意义的。下面我们仅对这一情况做简要的定性描述。

A particle with negative kinetic energy tends to the minimum of the action and therefore tends to achieve a maximal speed. If such a particle is free, it cannot accelerate, as this would violate energy conservation. Hence, a free ghost does not produce any harm to the environment, being isolated from it. However, in the case when there is an interaction of a ghost with healthy fields, the argument about energy conservation in a closed system does not work. Since any physical system tends to the state with minimal action, a ghost tends to accelerate, transmitting the extra positive energy to the healthy fields interacting with it, in the form of the quantum or classical emission of the corresponding particles. A systematic study of this situation at the quantum level has been given by Veltman in [58].

具有负动能的粒子会趋向作用量的最小值，因此倾向于达到最大速度。如果这类粒子是自由的，它便无法加速，因为这会违反能量守恒。因此，自由鬼粒子会与环境隔绝，不会对环境造成任何危害。然而，当鬼粒子与良性场存在相互作用时，封闭系统能量守恒的论点就不再成立。由于任何物理系统都趋向作用量最小的状态，鬼粒子会持续加速，同时将额外的正能以对应粒子的量子或经典辐射形式传递给和它相互作用的良性场。Veltman 在文献 [58] 中已经对这一情况在量子层面开展了系统研究。

Since gravitons are massless and the metric-dependent gravitational theories have non-polynomial interactions, a massive ghost always couples to an infinite amount of gravitons. This fact may lead to dramatic consequences. The energy conservation does not forbid a spontaneous creation of a massive ghost from the vacuum, even in the flat Minkowski space. It is clear that such a spontaneous creation of a ghost also implies that the corresponding amount of positive energy should be released with the creation of massless gravitons. As the mass of the ghost has the Planck magnitude, these gravitons have to accumulate with the Planck energy density.

由于引力子无质量，且依赖度规的引力理论具有非多项式相互作用，大质量鬼总会耦合无穷多引力子。这一事实可能引发严重后果。能量守恒并不禁止从真空中自发产生大质量鬼，哪怕在平直闵氏空间中也是如此。显而易见，这种鬼的自发产生意味着，对应量的正能量会随着引力子的产生被释放出来。由于鬼的质量为普朗克量级，这些引力子必然会以普朗克能量密度累积。

Assuming the existence of even a single real ghost, such a particle should accelerate, emitting and scattering gravitons. The magnitude of the energy of the ghost would increase, and hence, the energy of the created and scattered gravitons would increase too, without an upper bound for the emitted gravitational energy. After a while, the ghost would acquire an infinite amount of negative energy and start to emit an infinite amount of positive energy. It is clear that if some objects of this sort would be around, we would certainly know about it or, rather, we would feel it.

假设哪怕只存在一个真实的鬼粒子，这类粒子也会持续加速，发射并散射引力子。鬼粒子的能量绝对值会不断增大，因此产生和散射出的引力子能量也会随之增长，导致发射出的引力能不存在上限。一段时间后，鬼粒子将获得无穷多的负能量，进而开始发射出无穷多的正能量。很显然，如果周围存在这类物体，我们肯定能察觉它，更准确地说，我们早就直接感受到它的存在了。

Thus, the main theoretical problem is to explain why this dramatic scenario does not work. We have to say that, at the moment, there is no solution to this important problem. The solution could be, e.g., an explanation of why gravitons cannot agglomerate with the Planck density, but no mechanism for this has been formulated. Obviously, we have to assume that some kind of solution exists. Probably, it is related to the large mass of the massive ghost in higher-derivative gravity.

因此，核心理论问题在于解释为何这一极端情况并不会发生。必须说明的是，目前这一重要问题尚无解答。可能的解答方向例如解释引力子为何无法聚集达到普朗克密度，但至今尚未提出相关机制。显然，我们必须假设存在某类解法，它很可能与高阶引力中重幽灵的大质量性质相关。

The simplest way to preserve the unitarity is to admit the existence of a ghost. But, as we described above, this leads to the physically inconsistent output. Thus, one avoids ghosts by forming the in states only with gravitons. Owing to the interactions, ghost wakes up from the vacuum and emerge in the out states - that means the scattering matrix is not unitary [17] and we arrive at the contradiction.

保持幺正性最简单的方法是承认鬼场的存在。但正如我们上面所描述的，这会导致物理上不一致的结果。因此，人们仅用引力子构成初态来避免鬼场。由于相互作用，鬼场从真空“苏醒”并出现在末态中——这意味着散射矩阵不是幺正的 [17]，于是我们陷入了矛盾。

Historically, the main efforts in solving the problem of ghosts was related to the quantum aspects. In this respect we can start the list from the mentioned paper by Veltman [58], regardless it has nothing specific about quantum gravity. Soon after the seminal work of Stelle [17] with the proof of renormalizability of the fourth-derivative QG, there were first works about solving the problem of ghosts [59] and [60]. The main common idea of these works was that the loop corrections to the propagator (103) transform the unique massive pole into a pair of the complex conjugate poles, with the positions of these poles being gauge-fixing dependent [61]. In this case, one can prove the unitarity of the S -matrix, violated by the presence of massive ghosts [17]. Unfortunately, it was shown [62] that a definite conclusion on this issue can be taken only on the basis of exact knowledge of the dressed propagator of $h_{\mu\nu}$. For example, the one-loop corrections or the $1/N$ approximation is insufficient for solving the problem. It is interesting that starting from the six-derivative theory, one can provide the desirable features (pair of complex conjugate poles, for instance) already at the tree-level and, also, prove that the loop corrections do not change this structure. In this situation, one can use the optical theorem and prove the unitarity of the S -matrix in the Lee-Wick approach [63].

历史上，解决鬼问题的主要工作都和量子层面相关。就此而言，我们可以从维尔特曼上述论文 [58] 开始梳理，尽管这篇论文并未探讨量子引力的具体内容。在斯泰勒证明四导数量子引力可重整化的开创性工作 [17] 发表后不久，就出现了首批研究解决鬼问题的工作 [59] 与 [60]。这些工作的核心共同观点是: 对传播子 (103) 的圈修正会将单个质量极点转换为一对复共轭极点，且这些极点的位置依赖于规范固定 [61]。在这种情况下，我们可以证明因大质量鬼存在而被破坏的 S 矩阵的幺正性 [17]。遗憾的是，已有研究表明 [62]，只有在完全知晓 $h_{\mu\nu}$ 经修正后的传播子的前提下，才能对这个问题得出明确结论。例如，单圈修正或 $1/N$ 近似都不足以解决该问题。有趣的是，从六导数理论出发，我们已经可以在树图阶得到我们想要的性质 (比如一对复共轭极点)，同时还能证明圈修正不会改变这一结构。在这种情况下，我们可以利用光学定理，在李-维克框架下证明 S 矩阵的幺正性 [63]。

The problem with this solution of the ghost problem is that the Lee-Wick approach assumes that the scattering occurs between the asymptotical states, where in and out states describe the free particles. However, in gravity (and especially in its quantum version), the notion of free massive particle is not perfectly well defined, because any such particle produces a gravitational field, starts to interact with it, and, therefore, is not completely free. Therefore, the definite resolution of the ghost problem by means of the S -matrix does not look really promising, at least at the fundamental level.

这种解决鬼问题的方案存在一个问题: 李-维克方法假设散射发生在渐近态之间，其中入态和出态都描述自由粒子。然而在引力中 (尤其是量子引力中)，自由大质量粒子的概念并非定义良好，因为任何这类粒子都会产生引力场，并开始与引力场相互作用，因此并非完全自由。因此，借助 S 矩阵明确解决鬼问题的思路，至少在基础层面来看，并没有太好的前景。

What we can learn from all the quantum considerations is that the issues with ghosts described above would become impossible in the presence of a natural cutoff on the energy density of gravitons, such that this density never achieves the Planck order of magnitude. Then the gravitons cannot agglomerate to create a ghost and the S -matrix remains unitary. The problem is that there is no theoretical mechanism for such a cutoff. One can say that this is the main reason of why the problem of ghosts does not have a solution.

从所有这些量子研究中我们可以得到一个结论: 如果引力子的能量密度存在一个自然截断，使其永远无法达到普朗克量级，那么上述的鬼问题就不会存在。此时引力子无法聚集形成鬼，S 矩阵也始终保持幺正性。问题在于目前并不存在能实现这种截断的理论机制，这就是鬼问题至今无解的主要原因。

Stable Solutions in the Presence of Massive Ghosts

存在大质量鬼时的稳定解

Another aspect of the problem of ghosts is related to the stability of classical solutions. The most important issue related to massive ghosts is whether their presence can be compatible with the stability of the classical solutions of GR. As we mentioned above, in higher-derivative models of gravity, one typically meets the Planck suppression. As a result, classical solutions of GR represent high-quality approximations to the corresponding solutions in the presence of the higher-derivative terms. This logic can be successfully applied to the fourth-derivative theory and extended to the polynomial theories (34), if we assume that all massive parameters are of the Planck order of magnitude [37].

鬼问题的另一个方面和经典解的稳定性有关。与大质量鬼相关的核心问题是，它们的存在能否兼容广义相对论经典解的稳定性。正如我们前文提到，在高阶引力模型中，通常会存在普朗克压低效应。结果就是，即便存在高阶导数项，广义相对论的经典解仍是对应解的高质量近似。如果假设所有大质量参数都处于普朗克量级 [37]，这套逻辑可以成功应用于四阶导数理论，还可以推广到多项式理论 (34)。

The excellence in the approximation of the solution of GR does not necessarily mean the stability of this solution. In general, providing the stability of a gravitational solution under arbitrarily small perturbations (which do not have the symmetry of solutions themselves) may be not simple even in GR, for some gravitational backgrounds. Let us mention, for instance, the study of the stability of the Schwarzschild solution in GR [64, 65] (see also [66]). In the presence of the C^2 term, one can expect that the same solution will not be stable. Regardless of existing contradictions in the literature, in general, this expectation is confirmed [67, 68]. Owing to the high level of technical difficulty, this case will not be discussed here.

广义相对论解的近似表现优异，并不一定意味着该解具有稳定性。一般而言，即便在广义相对论中，对某些引力背景来说，要让引力解在任意微小扰动 (这些扰动不具备解本身的对称性) 下保持稳定也并非易事。例如，我们可以提及广义相对论中史瓦西解的稳定性研究 [64, 65] (另见文献 [66])。当存在 C^2 项时，可以预期同一解将不再稳定。尽管文献中现存矛盾，但总体而言这一猜想得到了证实 [67, 68]。由于技术难度极高，本文在此不讨论该情形。

From the technical side, it is much simpler to consider the stability of classical cosmological solutions in the presence of fourth-order terms. The advantage of this simpler case is that the physical interpretation of the results is relatively explicit. The analysis of [46] and after that in [47-49] was done for the fourth-derivative action (30) with anomaly-induced semiclassical corrections (see, e.g., [7] for the review). It turns out that these corrections do not change the main result. We shall explain this result omitting all technical details except the basic formulas.

从技术层面来看，研究存在四阶项时经典宇宙学解的稳定性要简单得多。该简化情形的优势在于，所得结果的物理解释相对清晰。文献 [46] 率先开展了相关分析，后续文献 [47-49] 针对带反常诱导半经典修正的四阶导数作用量 (30) 展开研究 (综述可参见例如文献 [7])。结果表明，这些修正并未改变核心结论。我们将仅列出基础公式，省略所有技术细节，对该结论进行说明。

The stability we need to explore is related to the presence of massive spin-2 ghost degrees of freedom, which means the transverse and traceless modes of the metric perturbation on the homogeneous and isotropic, cosmological background. According to the theory of cosmological perturbations (see, e.g., [69-71]), the background cosmological metric with tensor perturbations is

我们需要探究的稳定性与有质量自旋 2 鬼自由度的存在相关，后者是指均匀各向同性宇宙背景下度量微扰的横向无迹模式。根据宇宙微扰理论 (例如参见文献 [69-71])，带张量微扰的背景宇宙度量为

$$ds^2 = a^2(\eta)\left[d\eta^2 - \left(\gamma_{ij} + h_{ij}\right) dx^i \, dx^j\right], \tag{126}$$

where η is the conformal time, $a(\eta)$ corresponds to a background cosmological solution, and we imposed the synchronous coordinate condition $h_{\mu 0} = 0$. Furthermore, for the sake of simplicity we consider the flat space geometry $k = 0$ (hence, $\gamma_{ij} = \eta_{ij}$) and set the cosmological constant to vanish, $\Lambda = 0$.

其中 η 为共形时间，$a(\eta)$ 对应背景宇宙学解，我们采用了同步坐标条件 $h_{\mu 0} = 0$。此外，为简化计算我们考虑平直空间几何 $k = 0$ (因此有 $\gamma_{ij} = \eta_{ij}$)，并令宇宙学常数为零，即 $\Lambda = 0$。

Since we are interested in the gravitational wave dynamics, it is sufficient to retain only the traceless and transverse parts of h_{ij} , which are the purely tensor modes, by imposing

由于我们关注引力波动力学，仅需通过施加条件保留 h_{ij} 的无迹部分与横波部分即可，这两部分对应纯张量模

$$\partial_i h^{ij} = 0, \; h_{kk} = 0. \tag{127}$$

As before, we do not need to write indices and set $h = h^{ij}$.

和之前一样，我们不需要写出指标，设 $h = h^{ij}$。

The Lagrange equation for $h(t)$, in terms of physical time, has the form [72]

以物理时间表示的 $h(t)$ 的拉格朗日方程形式如下 [72]

$$\begin{aligned}
&\frac{1}{3} h^{(\mathrm{IV})} + 2H h^{(\mathrm{III})} + \left(H^2 - \frac{\lambda M_{\mathrm{P}}^2}{16\pi}\right)\ddot{h} + \frac{2}{3}\left(\frac{1}{4}\frac{\nabla^4 h}{a^4} - \frac{\nabla^2 \ddot{h}}{a^2} - H\frac{\nabla^2 \dot{h}}{a^2}\right) \\
&-\left(H\dot{H} + \ddot{H} + 6H^3 + \frac{3\lambda M_{\mathrm{P}}^2 H}{16\pi}\right)\dot{h} + \left[\frac{\lambda M_{\mathrm{P}}^2}{16\pi} + \frac{4}{3}\left(\dot{H} + 2H^2\right)\right]\frac{\nabla^2 h}{a^2} \\
&-\left[24\dot{H}H^2 + 12\dot{H}^2 + 16H\ddot{H} + \frac{8}{3}H^{(\mathrm{m})} + \frac{\lambda M_{\mathrm{P}}^2}{8\pi}\left(2\dot{H} + 3H^2\right)\right] h = 0.
\end{aligned} \tag{128}$$

The contribution of the fourth-derivative terms depends on the unique parameter λ from the action (30). The reason is that the Gauss-Bonnet combinations do not affect the equations of motion and another invariant is R^2 , which contributes to the equation for the conformal factor $a(t)$, but does not affect the propagation of the tensor mode.

四阶导数项的贡献依赖于作用量 (30) 中的唯一参数 λ。原因在于高斯-博内组合不影响运动方程，另一个不变量为 R^2 ，它对标度因子 $a(t)$ 的方程有贡献，但不影响张量模的传播。

The next step is to make the Fourier transformation for the space coordinates

下一步是对空间坐标做傅里叶变换

$$h_{\mu\nu}(\mathbf{r},t) = \int \frac{d^3k}{(2\pi)^3} h_{\mu\nu}(\mathbf{k},t)\, e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{129}$$

One can treat the wave vector $\mathbf{k}$ as a constant and hence will be interested only in the time evolution of the perturbation $h_{\mu\nu}(\mathbf{k},t)$. The validity of such a treatment is restricted to the linear perturbations, but this is what we need now. In this way, the complicated partial differential equation (128) is reduced to the much simpler ordinary differential equation for each of the individual modes.

我们可以将波矢 $\mathbf{k}$ 视为常数，因此仅需要研究微扰 $h_{\mu\nu}(\mathbf{k},t)$ 的时间演化。该处理仅在线性微扰框架下成立，而这正是我们目前所需。通过这种处理，复杂的偏微分方程 (128) 被约化为每个单独模式对应的简单得多的常微分方程。

Using the notation $h = h(t,\mathbf{k}) = h(t,k)$, the equation has the form

使用记号 $h = h(t,\mathbf{k}) = h(t,k)$ ，该方程具有如下形式

$$\begin{aligned} &h^{(\mathrm{IV})} + 6Hh^{(\mathrm{III})} + \left(3H^2 - \frac{3\lambda M_{\mathrm{P}}^2}{16\pi}\right)\ddot{h} + \left(\frac{1}{2}\frac{k^4}{a^4}h + \frac{2k^2}{a^2}\ddot{h} + \frac{2k^2}{a^2}H\dot{h}\right) \\ &-3\left(H\dot{H} + \ddot{H} + 6H^3 + \frac{3\lambda M_{\mathrm{P}}^2 H}{16\pi}\right)\dot{h} - \left[\frac{3\lambda M_{\mathrm{P}}^2}{16\pi} + 4\left(\dot{H} + 2H^2\right)\right]\frac{k^2}{a^2}h \\ &-\left[72\dot{H}H^2 + 36\dot{H}^2 + 48H\ddot{H} + 8H^{(\mathrm{III})} + \frac{3\lambda M_{\mathrm{P}}^2}{8\pi}\left(2\dot{H} + 3H^2\right)\right]h = 0, \end{aligned}$$

(130)

where $k = |\mathbf{k}|$ is the frequency of the massless field. Finally, the initial conditions for the perturbations will be chosen according to the quantum fluctuations of free fields. The spectrum is identical to that of a scalar quantum field in Minkowski space [2],

其中 $k = |\mathbf{k}|$ 是无质量场的频率。最后，微扰的初始条件将按自由场的量子涨落选取，该谱与闵氏空间中标量量子场的谱一致 [2],

$$h(x,\eta) = h(\eta)\, e^{\pm i\mathbf{k}.\mathbf{r}},\ h(\eta) \propto \frac{e^{\pm ik\eta}}{\sqrt{2k}}. \tag{131}$$

As before, η is conformal time. The transition to the physical time is $a(\eta)\,d\eta = dt$. The normalization constant in (131) is irrelevant for the linear perturbations.

和之前一样，η 是共形时间，转换为物理时间为 $a(\eta)\,d\eta = dt$ 。式 (131) 中的归一化常数对线性微扰无关紧要。

The possible instabilities can be explored using Eq. (130). According to the known mathematical theorems about the stability of the fixed points of differential equations, linear stability guarantees nonlinear (at least perturbative) stability for sufficiently small perturbations. In the present case, this point was confirmed in [48, 49] for the Bianchi-I metric, which is a reduced form of the perturbation for $\mathbf{k} = 0$.

> 我们可以利用式 (130) 探究可能的不稳定性。根据微分方程不动点稳定性的已有数学定理，对于足够小的微扰，线性稳定性可保证非线性 (至少是微扰论层面的) 稳定性。在当前案例中，文献 [48, 49] 已针对毕安基-I 度规证实了这一结论，该度规是 $\mathbf{k} = 0$ 微扰的约化形式。

The main qualitative result of the numerical analysis of Eq. (130) performed in [46,47] is that the linear stability in the fourth-derivative model is possible, but the two conditions should be fulfilled. First of all, the frequency k should be essentially smaller than the Planck mass. The threshold value for k slightly depends on the type of cosmological solution (dominated by radiation, dust, or cosmological constant), but, in all cases for $k < 0.1M_P$, there is no growth of $h\,(t, k)$, while such a growth is evident starting from $k \approx 0.6M_P$. This requirement exactly corresponds to our expectation that one needs a Planck density of gravitons to wake up the ghost from the vacuum. For a small frequency k , the ghost remains as a virtual mode and cannot be created from vacuum to become a real particle. Let us remember that the creation of a ghost from the vacuum requires positive-energy gravitons with the Planck energy density (Planck energy in the space volume of a cube of the Planck-scale Compton wavelength). If the frequency of the gravitational wave is insufficient, the ghost is not created, and there is no instability.

> 文献 [46,47] 对式 (130) 进行数值分析得到的主要定性结论是: 四阶导数模型中可以存在线性稳定性，但需要满足两个条件。首先，频率 k 必须远小于普朗克质量。k 的阈值对宇宙学解的类型 (辐射主导、尘埃主导或宇宙学常数主导) 依赖很弱，但在所有情况下，当满足 $k < 0.1M_P$ 时，$h\,(t, k)$ 都不会增长；而当满足 $k \approx 0.6M_P$ 时，这种增长就会显现。这一要求完全符合我们的预期: 即需要普朗克密度的引力子才能将鬼从真空中唤醒。当频率 k 较小时，鬼始终保持为虚模式，无法从真空中产生成为实粒子。别忘了，从真空中产生一个鬼需要具有普朗克能量密度的正能引力子 (在普朗克尺度康普顿波长立方空间体积中能量为普朗克能量)。如果引力波频率不足，鬼就无法产生，也就不会出现不稳定性。

Second, the signs of both parameters λ and κ^2 should be positive. For a negative sign of the calibrated Newton constant κ^2 , the ghost becomes also a tachyon and there is no stability for any frequency. The negative sign of λ means the graviton becomes a ghost and the massive particle is normal. Then there is no threshold for creating a ghost and we observe instabilities at all frequencies. And this is exactly what was observed in the numerical analysis in [46]. It worth noting that, in the previous subsection, we saw that $\lambda > 0$ is a condition of stability in the flat spacetime. Now we can see that this is confirmed by the analysis of stability on the cosmological background. Thus, in what follows we assume that λ and κ^2 are both positive.

> 其次，参数 λ 和 κ^2 的符号都必须为正。如果校准后的牛顿常数 κ^2 为负，鬼也会成为快子，且任何频率下都不存在稳定性。如果 λ 为负，则意味着引力子变成鬼，而大质量粒子是正常粒子。此时产生鬼不存在阈值，我们在所有频率下都会观测到不稳定性。这正是文献 [46] 数值分析中观测到的结果。值得注意的是，在上一小节中我们已经知道，$\lambda > 0$ 是平坦时空下的稳定性条件。现在我们看到，宇宙学背景下的稳定性分析也证实了这一点。因此，在下文中我们假设 λ 和 κ^2 均为正。

As we mentioned already a few times, the solution of the ghost problem (and, consequently, of the QG problem in general) requires an explanation of why gravitons cannot accumulate with the Planck energy density (One can find the discussions of other implications of this unknown physical principle in [73].). Intuitively, it is easier to accept that such accumulation may occur only when the background metric describes an intensive gravitational field, as in the early Universe.

正如我们已经多次提到的，要解决鬼问题 (进而解决整体量子引力问题)，就需要解释为什么引力子无法累积达到普朗克能量密度 (关于这一未知物理原理的其他推论，可见文献 [73] 中的讨论)。凭直觉来看，我们更容易接受只有当背景度量描述强引力场时 (比如早期宇宙)，这种累积才会发生。

In this respect, an interesting thing happens when the frequency k is greater than the energy threshold but the external cosmological background is described by a strong gravity. The last means that the Hubble parameter has a large value and the Universe is rapidly expanding. It turns out that there is a very fast growth of $h(t,k)$ but such an explosion of the perturbations does not last for a long period of time. To understand why this happens, one can take a look at the main equation, (130). The frequency k enters this equation in the combination $q = k/a(t)$. For a sufficiently fast expansion of the Universe, the explosive growth of the perturbation lasts only until the magnitude of q becomes smaller than the energy threshold. After that, the amplitude of the perturbations vanish exponentially. Thus, the perturbations do not violate the cosmological principle, i.e., the Universe remains homogeneous and isotropic at the large scale and the effect of ghosts does not contradict the observations.

在这方面，当频率 k 大于能量阈值，且外部宇宙学背景由强引力描述时，会出现一个有趣的情况。后者意味着哈勃参数数值很大，宇宙在快速膨胀。结果就是 $h(t,k)$ 会出现极快的增长，但这种微扰爆发不会持续很长时间。要理解其中原因，我们可以来看主方程 (130): 频率 k 以组合 $q = k/a(t)$ 的形式出现在方程中。对于宇宙足够快速膨胀的情况，微扰的爆发式增长只会持续到 q 的大小小于能量阈值为止。在这之后，微扰的振幅会指数衰减。因此，微扰不会破坏宇宙学原理，即宇宙在大尺度上仍然保持均匀各向同性，鬼的效应并不与观测矛盾。

To end this subsection, let us stress that the result of [46,47] and [48,49] cannot be interpreted as a solution of the problem of massive ghosts. In our opinion, it should be regarded as a hint to the direction where such a solution can be found.

在本小节最后，我们需要强调，文献 [46,47] 和 [48,49] 得出的结果并不能解决大质量鬼问题。在我们看来，这一结果仅仅为找到该问题的解决方向提供了线索。

Effective Approach to the Problem of Ghosts

引力子问题的有效方法

The effective approach to QG is a subject of a special section of our handbook, so it makes no sense to go into details of this approach here. Let us just briefly explain what is conventionally understood as an effective solution to the problem of massive ghosts, as introduced by Simon in [74] and elaborated further in [75].

量子引力的有效方法是本手册一个专门章节的主题，因此在此处详述该方法没有意义。我们仅简要说明，何为大规模幽灵问题的常规有效解——这一方案由西蒙在文献 [74] 中提出，并在文献 [75] 中得到进一步拓展。

The proposal in [74] is to consider the Einstein equations as the basic gravitational theory, regarding all higher-derivative terms in the gravitational action and the respective dynamical equations as a small perturbation. According to this treatment, the gravitational theory should be described by the two physical degrees of freedom of GR by definition - independent of what the action of the theory is and the form of the quantum corrections to this action. The propagator of the quantum metric $h_{\mu\nu}$ is derived from Einstein's gravity, and no corrections can produce additional poles in this propagator. By construction, there cannot be any kind of massive ghosts, and hence, there are no problems with unitarity and instabilities at the classical or quantum levels.

> 文献 [74] 的方案是将爱因斯坦引力方程视作基础引力理论，把引力作用量和对应动力学方程中的所有高阶导数项视为小微扰。根据这套处理方式，引力理论按定义应当仅由广义相对论的两个物理自由度描述——无论该理论的作用量形式如何，也无论该作用量的量子修正采用何种形式。量子度规 $h_{\mu\nu}$ 的传播子由爱因斯坦引力推导得出，任何修正都不会在该传播子中产生额外极点。根据这套构造，不可能存在任何形式的大规模幽灵，因此在经典或量子层面都不存在幺正性和不稳定性问题。

This solution certainly looks mathematically correct and efficient. On the other hand, there are serious problems with its consistency, especially at the high-energy scale, where the fourth-derivative terms gain the magnitude comparable to the GR action. On top of that, there is a problem with the uniqueness of the procedure. For example, one can modify this effective approach to include an R^2 term, or any $f(R)$ term, into the main part of the action, because these terms do not produce a ghost. The same concerns many other terms, e.g., all $\mathcal{O}\left(R^3 \ldots\right)$ ' s. At the same time, it should be strictly forbidden to do the same with the $R\Box R$ -term, which produces a scalar ghost. As it was discussed in [37] and also in [7], the analogy with QED and the standard resolution of the problem of "runaway solutions" is not convincing. So, all the scheme looks as an ad hoc procedure, without the physical background. It is as saying that we do not like ghosts and will therefore forbid them. If we follow the same approach in other branches of QFT, it is perfectly well possible to modify any theory in a way we like and provide the predictions we like. Despite this may look a universal solution of all the problems, the theories created in this way would not be reliable or, better say, would not provide reliable predictions.

> 这套解法在数学上无疑是正确且高效的。但另一方面，它的一致性存在严重问题，在高能标度下尤其如此——四阶导数项在高能标度的量级与广义相对论作用量相当。此外，这套处理流程还存在唯一性问题。例如，可以对该有效方法进行修改，将一个 R^2 项，或任意 $f(R)$ 项纳入作用量的主体部分，因为这些项不会产生幽灵。许多其他项也是同理，例如所有 $\mathcal{O}\left(R^3 \ldots\right)$ 项。同时，对会产生标量幽灵的 $R\Box R$ 项，必须严格禁止这么处理。正如文献 [37] 和文献 [7] 中讨论的，将其与量子电动力学类比、用“逃逸解”问题的标准解法来解释并不令人信服。因此，整个方案看起来就是一个特设流程，没有物理基础。这就相当于说我们不想要幽灵，所以直接禁止它们存在。如果我们在量子场论的其他分支也采用这套方法，完全可以随心所欲修改任何理论、得到想要的任何预测。即便这看起来能普适解决所有问题，用这种方式得到的理论并不靠谱，或者说，无法给出可靠的预测。

However, if we restrict the area of application of QG to the energies essentially lower than the Planck threshold, the described effective approach becomes a normal feature of the theory that can be fixed only by the observations and/or experiments. In this case, the approach of [74,75] becomes equivalent to the one described in the previous subsection. The results on the stability of the cosmological background of [46] show that, in the IR, one can trade using GR as a basic theory "by definition" to the restrictions on the initial seeds of the tensor mode of cosmological perturbations.

不过，如果我们将量子引力的适用范围限制在远低于普朗克能标的能量区间，上述有效方法就会成为理论的合理特征，只能通过观测和/或实验确定。在这种情况下，文献 [74,75] 的方法就等价于上一小节描述的方法。文献 [46] 关于宇宙背景稳定性的研究结果表明，在红外区，我们可以将“按定义将广义相对论作为基础理论”替换为对宇宙微扰张量模初始种子的限制。

To conclude this section, we have to say that the problem of ghosts is unsolved, at least if we do not restrict the applicability of QG to the low-energy (IR) domain. However, there are certain glues about the directions in which the solution may be found. It seems that we need a new physical principle forbidding the concentration of gravitons with the Planck densities. This means, at the Planck frequencies gravity should dramatically change. Such a change may be because of the nonlocalities in the action, but the problem may require a more complicated solution. At the UV, ghosts may be generated from the vacuum. In the preprint [76], Hawking made a hypothesis that, in this situation, the QFT approach should be modified, taking into account that the ghost is not an independent particle, but appears paired with the graviton. Thinking along this line, one can expect to find a solution by working with bound states or condensates, including ghosts and maybe some normal degrees of freedom, e.g., in the framework of the superrenormalizable QG.

最后总结本节: 至少在我们不将量子引力的适用性限制在低能 (红外) 区域的情况下，幽灵问题仍未解决。不过，关于可能找到解的研究方向，已经有了一些线索。我们似乎需要一条全新的物理原理，禁止普朗克密度下的引力子聚集。这意味着，在普朗克频率下引力会发生剧变。这种变化可能源自作用量中的非定域性，但这个问题可能需要更复杂的解决方案。在紫外区，幽灵可以从真空中产生。在预印本 [76] 中，霍金提出了一个假说: 在这种情况下，应当修改量子场论框架，因为幽灵不是独立粒子，而是和引力子成对出现。沿着这个思路，我们有望通过研究束缚态或凝聚态找到解决方案，其中包含幽灵和可能的一些正常自由度，例如在超可重整化量子引力的框架下。

Gauge-Fixing Dependence Using General Formalism

利用通用形式体系讨论规范固定依赖性

Starting from this point, in this and the next sections, we discuss the loop corrections in the models of QG. This is an extensive subject and it is traditionally one of the most worked out parts of QG. Obviously, all of it cannot be settled into a short review, so we shall discuss only two particular aspects. Namely, we perform an important general analysis of the gauge and parametrization dependence of the loop corrections in QG and show the derivation of loop corrections in the simplest case of quantum GR in the simplest gauge and parametrization of the quantum metric field, i.e., repeat the main part of the paper by 't Hooft and Veltman [32]. Another chapter of this section is devoted to the divergences in the fourth-derivative QG model.

从这里开始，我们将在本节和下一节讨论量子引力模型中的圈修正。这是一个内容广泛的主题，传统上也是量子引力中研究最多的领域之一。显然，短短一篇综述不可能涵盖全部内容，因此我们仅讨论两个特定方面: 我们将对量子引力中圈修正的规范依赖性与参数化依赖性开展重要的通用分析，展示最简单规范与量子度规场参数化下最简量子广义相对论情形中圈修正的推导，也就是重现特霍夫特和韦尔特曼论文 [32] 的核心内容。本节另一部分内容专门讨论四阶导数量子引力模型中的发散问题。

In this section, we show how the general statement about the on-shell gauge-fixing and parametrization independence of the effective action can be used in different models of QG. The practical applications described below were introduced in [24] and, later on, formulated in a more explicit form in [77] and [78].

在本节中，我们将说明有效作用量的在壳规范固定独立性与参数化独立性这一通用结论，如何应用于不同的量子引力模型。下文介绍的实际应用最早由文献 [24] 提出，之后文献 [77] 和 [78] 给出了更明确的表述。

Gauge-Fixing Dependence in Quantum GR

量子广义相对论中的规范固定依赖性

Consider the gauge-fixing and parametrization dependence in QG based on GR. We consider the nonzero cosmological constant for generality and also because an interesting application to the on-shell renormalization group [24].

我们来研究基于广义相对论的量子引力中的规范固定与参数化依赖性。为了更具普适性，同时也是为了将其有趣地应用于壳上重整化群 [24]，我们考虑非零宇宙学常数。

Let us start from some historic note and references. The subject was pioneered in the paper [79], where the calculations for the two-parameter gauge were performed using Feynman diagrams. Even more general diagram-based calculations in the nonminimal gauge were done in [80]. The use of the heat-kernel methods required the generalized Schwinger-DeWitt technique [25], and the results were applied to quantum GR in [81]. On the other hand, in [82] it was noted that one can simplify things by exploring the parametrization ambiguity instead of the minimal gauge fixing. The most general version of such a calculation [83] used the background field method with the parametrization

我们先从历史记录和参考文献开始讲起。该研究方向由文献 [79] 开创，作者在该文中使用费曼图完成了双参数规范的计算。文献 [80] 完成了非最小规范中更具一般性的基于图的计算。热核方法的使用需要广义施温格-德维特技术 [25]，文献 [81] 将相关结果应用到了量子广义相对论中。另一方面，文献 [82] 指出，我们可以通过研究参数化歧义而非最小规范固定来简化问题。这类计算最一般的版本 [83] 使用背景场方法结合参数化

$$g_{\alpha\beta} \to g'_{\alpha\beta} = e^{2\kappa r\sigma}\left[g_{\alpha\beta} + \kappa\left(\gamma_1\phi_{\alpha\beta} + \gamma_2\phi g_{\alpha\beta}\right)\right.$$

$$\left.+\kappa^2\left(\gamma_3\phi_{\alpha\rho}\phi^{\rho}_{\beta} + \gamma_4\phi_{\rho\omega}\phi^{\rho\omega}g_{\alpha\beta} + \gamma_5\phi\phi_{\alpha\beta} + \gamma_6\phi^2 g_{\alpha\beta}\right)\right] \tag{132}$$

where $g_{\alpha\beta}$ is the background metric and $\phi_{\alpha\beta}$ and σ are the quantum fields. Furthermore, the trace is defined as $\phi = \phi^{\mu}_{\mu}$ and the indexes are lowered and raised with the background metric $g_{\alpha\beta}$ and with its inverse $g^{\alpha\beta}$. For the one-loop effects, Eq. (132) is the generalized version of the simplest parametrization (9), as it includes arbitrary coefficients $\gamma_{1,2,\ldots,6}$ and r , which parameterize the choice of the quantum variables. An important detail is that, since the one-loop divergences are defined only by the bilinear in the quantum field part of the action, (132) is the most general parametrization at the one-loop order. The next point is that, for any choice of $\gamma_{1,2,\ldots,6}$ and r , one can choose the gauge-fixing parameters α and β such that the bilinear form of the total action with the S_{gf} -term (18) is a minimal operator. This approach enables one to verify the general statements about gauge-fixing and parametrization dependence in a relatively economic way, avoiding working with the nonminimal operators.

其中 $g_{\alpha\beta}$ 是背景度规，$\phi_{\alpha\beta}$ 和 σ 是量子场。此外，迹定义为 $\phi = \phi_\mu^\mu$，指标由背景度规 $g_{\alpha\beta}$ 及其逆 $g^{\alpha\beta}$ 升降标。对于单圈效应，式 (132) 是最简单参数化 (9) 的推广形式，因为它包含任意系数 $\gamma_{1,2,\ldots,6}$ 和 r，这两个系数用来参数化量子变量的选择。一个重要细节是，由于单圈发散仅由作用量中量子场部分的双线性项定义，因此 (132) 是单圈阶最一般的参数化。接下来，对于任意 $\gamma_{1,2,\ldots,6}$ 和 r 的选择，我们可以选择规范固定参数 α 和 β，使得带 S_{gf} 项 (18) 的总作用量的双线性形式是最小算子。该方法让我们可以用相对经济的方式验证关于规范固定和参数化依赖性的一般性结论，避免处理非最小算子。

In this review, we will not go into details of the practical calculations, which can be seen in the mentioned original works, but follow [78,83] and [7] to explore all the mentioned dependencies in the general framework.

在本综述中，我们不会展开介绍实际计算的细节，相关细节可以在上述原始文献中找到，我们将遵循 [78,83] 和 [7] 的内容，在通用框架下研究所有上述依赖性。

The classical equations of motion corresponding to the Einstein-Hilbert action with the cosmological constant term (22) are (we ignore the irrelevant factor of κ^2)

对应带宇宙学常数项 (22) 的爱因斯坦-希尔伯特作用量的经典运动方程为 (我们忽略了不相关的 κ^2 因子)

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g_{\mu\nu}} = R^{\mu\nu} - \frac{1}{2}\left(R + 2\Lambda\right) g^{\mu\nu}. \tag{133}$$

The general statement about gauge-fixing and parametrization independence on-shell can be used together with the locality of the divergent part of the effective action. The power counting tells us that this divergence has the form

关于规范固定和参数化的壳上无关性的一般性结论，可以结合有效作用量发散部分的局域性一起使用。量纲分析表明，该发散具有如下形式

$$\Gamma^{(1)}_{\text{div}} = \frac{1}{\varepsilon} \int d^4x \sqrt{-g} \left\{ c_1 R^2_{\mu\nu\alpha\beta} + c_2 R^2_{\alpha\beta} + c_3 R^2 + c_4 \Box R + c_5 R + c_6 \right\}, \tag{134}$$

where $\varepsilon = (4\pi)^2 (n-4)$ is the regularization parameter and $c_{1,2,\ldots,6}$ are some coefficients. Our purpose is to explore how these coefficients depend on the parametrization and gauge-fixing choices.

其中 $\varepsilon = (4\pi)^2 (n-4)$ 是正则化参数，$c_{1,2,\ldots,6}$ 是若干系数。我们的目的是研究这些系数如何依赖于参数化和规范固定的选择。

We denote α_i the full set of arbitrary parameters characterizing the gauge fixing and parametrization of the quantum metric. The special values α_i^0 of these parameters correspond to some fixed choice, e.g., to those in [32]. The α_i -related ambiguities in $\Gamma^{(1)}_{\text{div}}$ do not violate the locality of this expression. Taking this into account, the on-shell universality tells us that

我们用 α_i 标记描述量子度规的规范固定和参数化的全部任意参数。这些参数的特殊值 α_i^0 对应某一固定选择，例如文献 [32] 中的选择。$\Gamma_{\text{div}}^{(1)}$ 中与 α_i 相关的歧义不会破坏该表达式的局域性。考虑到这一点，壳上普适性告诉我们

$$\delta\Gamma_{\text{div}}^{(1)} = \Gamma_{\text{div}}^{(1)}\left(\alpha_i\right) - \Gamma_{\text{div}}^{(1)}\left(\alpha_i^0\right) \tag{135}$$

$$= \frac{1}{\varepsilon}\int d^4x\sqrt{-g}\left(b_1 R_{\mu\nu} + b_2 R g_{\mu\nu} + b_3 \Lambda g_{\mu\nu} + b_4 g_{\mu\nu}\Box + b_5 \nabla_\mu \nabla_\nu\right)\varepsilon^{\mu\nu},$$

where the new parameters $b_{1,2,\ldots,5}$ in (135) depend on α_i and the explicit form of the dependence can be seen only from the real calculations. However, one can draw relevant conclusions directly from (135). In the simplest case of $\Lambda = 0$, this formula tells us that only the Gauss-Bonnet counterterm $\int E_4$ cannot be set to zero by choosing α_i . This is exactly the result that was discovered by direct calculation in [79]. The S -matrix for the gravitational perturbations corresponds to the on-shell limit of the effective action, and thus, it is finite.

其中 (135) 式中的新参数 $b_{1,2,\ldots,5}$ 依赖于 α_i ，该依赖的显式形式只能通过实际计算得到。但我们可以直接从 (135) 得到相关结论。在 $\Lambda = 0$ 的最简单情况，该公式告诉我们，只有高斯-博内 counterterm $\int E_4$ 无法通过选择 α_i 置零。这正好是文献 [79] 中通过直接计算得到的结果。引力微扰的 S 矩阵对应有效作用量的壳上极限，因此它是有限的。

In the general theory, with $\Lambda \neq 0$, we note that the parameter b_5 has no effect on divergences because of the third Bianchi identity $\nabla_\mu G^\mu{}_\nu = 0$ and that $\nabla_\mu \Lambda = 0$. Thus, there is a four-parameter $b_{1,2,3,4}$ ambiguity for the six existing coefficients $c_{1,2,\ldots 6}$. Therefore, only two combinations of these six coefficients can be expected to be gauge-fixing and parametrization independent. Obviously, one of these combinations is the coefficient of $\int E_4$ defined in (29). This directly follows from the fact that the Λ -term cannot affect the four-derivative divergences, as we know from the power counting.

在一般理论中，对于 $\Lambda \neq 0$ ，我们注意到由于第三比安基恒等式 $\nabla_\mu G^\mu{}_\nu = 0$ ，参数 b_5 对发散没有影响，且 $\nabla_\mu \Lambda = 0$ 。因此，对于六个现有系数 $c_{1,2,\ldots 6}$ ，存在一个四参数 $b_{1,2,3,4}$ 的不确定性。因此，仅能预期这六个系数中的两个组合是规范固定和参数化无关的。显然，其中一个组合是 (29) 中定义的 $\int E_4$ 的系数。这可直接由下述事实推出: 正如我们从幂次计数得知的，Λ 项不会影响四阶导数发散。

Let us find the second combination of the parameters. A simple calculation using (135) shows that the coefficients in the expression (134) vary according to

我们来寻找参数的第二个组合。利用 (135) 进行的简单计算表明，(134) 表达式中的系数按照下式变化

$$c_1 \to c_1,\ c_2 \to c_2 + b_1,\ c_3 \to c_3 - \left(b_2 + \frac{1}{2}b_1\right),$$

$$c_4 \to c_4 - b_4,\ c_5 \to c_5 - \left(b_1 + 4b_2 + b_3\right)\Lambda,\ c_6 \to c_6 - 4b_3\Lambda^2.$$

(136)

It is an easy exercise to show that the two gauge-fixing and parametrization invariants which do not change under the transformations of $c_{1,2,\ldots,5}$ in (136) are

不难证明，在 (136) 中 $c_{1,2,\ldots,5}$ 的变换下保持不变的两个规范固定和参数化不变量为

$$c_1 \quad \text{and} \quad c_{\text{inv}} = c_6 - 4\Lambda c_5 + 4\Lambda^2 c_2 + 16\Lambda^2 c_3. \tag{137}$$

The last observation is that the on-shell expressions for the classical action and divergences have the forms

最后注意，经典作用量和发散的壳上表达式形式如下

$$S\big|_{\text{onshell}} = \frac{2\Lambda}{\kappa^2}\int d^4x\sqrt{-g},$$

$$\Gamma^{(1)}_{\text{div}}\Big|_{\text{onshell}} = \frac{1}{\varepsilon}\int d^4x\sqrt{-g}\,\{c_1 E_4 + c_{\text{inv}}\}. \tag{138}$$

In these two functionals there are only invariant quantities. This feature forms the basis of the so-called on-shell renormalization group equation, to be discussed below. An additional small detail is that c_{inv} does not change if we replace E_4 in (138) by the square of the Riemann tensor.

这两个泛函中仅存在不变量。这一性质是下文将要讨论的所谓壳上重整化群方程的基础。一个额外的小细节是: 若我们将 (138) 中的 E_4 替换为黎曼张量的平方，c_{inv} 保持不变。

Gauge-Fixing Dependence in Higher-Derivative Models

高阶导数模型中的规范固定依赖性

Adding more derivatives in the classical action, the gauge-fixing and parametrization dependence in the divergent part of the effective action becomes smaller. Among all higher-derivative models of QG, the unique nontrivial example is the fourth-derivative model (30). Let us first consider this model following [78].

在经典作用量中加入更多导数后，有效作用量发散部分的规范固定和参数化依赖性会变小。在所有量子引力高阶导数模型中，唯一非平凡的例子是四阶导数模型 (30)。我们首先遵循文献 [78] 讨论该模型。

In the four-derivative theory, the formula analogous to (135) has the form

在四阶导数理论中，类似 (135) 的公式形式为

$$\Gamma^{(1)}_{\text{div}}(\alpha_i) - \Gamma^{(1)}_{\text{div}}(\alpha_i^0) = \frac{1}{\varepsilon}\int d^4x\sqrt{-g}\,f_{\mu\nu}\varepsilon^{\mu\nu}_{(4)}, \tag{139}$$

where $f_{\mu\nu} = f_{\mu\nu}(\alpha_i)$ is an unknown tensor depending on α_i and

其中 $f_{\mu\nu} = f_{\mu\nu}(\alpha_i)$ 是依赖于 α_i 的未知张量，且

$$\varepsilon^{\mu\nu}_{(4)} = \frac{1}{\sqrt{-g}} \frac{\delta S_{HD}}{\delta g_{\mu\nu}} \tag{140}$$

are the equations of motion for the fourth-derivative gravity.

是四阶导数引力的运动方程。

To find $f_{\mu\nu}$, let us remember that the fourth-derivative quantum gravity is a renormalizable theory. Therefore, all three of the expressions $\Gamma^{(1)}_{\text{div}}\left(\alpha^0_i\right), \Gamma^{(1)}_{\text{div}}\left(\alpha_i\right)$, and $\varepsilon^{\mu\nu}_{(4)}$ have dimension 4, as the classical action. Since the divergencies in (139) are local functionals, $f_{\mu\nu}$ is a dimensionless tensor. Then the only possible choice is

为了找到 $f_{\mu\nu}$ ，我们知道四阶导数量子引力是可重整化理论。因此，表达式 $\Gamma^{(1)}_{\text{div}}\left(\alpha^0_i\right), \Gamma^{(1)}_{\text{div}}\left(\alpha_i\right)$ 和 $\varepsilon^{\mu\nu}_{(4)}$ 都和经典作用量一样量纲为 4。由于 (139) 中的发散是局部泛函，$f_{\mu\nu}$ 是无量纲张量。那么唯一可能的选择是

$$f_{\mu\nu}\left(\alpha_i\right) = g_{\mu\nu} f\left(\alpha_i\right), \tag{141}$$

where $f\left(\alpha_i\right)$ is an arbitrary (can be defined only by explicit calculations) dimensionless function of the set of parameters of gauge fixing and metric parametrization. Thus, the gauge/parametrization dependence of the divergent part of effective action is controlled by the 'conformal shift' of the classical action

其中 $f\left(\alpha_i\right)$ 是规范固定和度规参数化参数集合的任意无量纲函数 (只能通过显式计算确定)。因此，有效作用量发散部分的规范/参数化依赖性由经典作用量的“共形平移”控制

$$\Gamma^{(1)}_{\text{div}}\left(\alpha_i\right) - \Gamma^{(1)}_{\text{div}}\left(\alpha^0_i\right) = f\left(\alpha_i\right) \int d^4x g_{\mu\nu} \frac{\delta S}{\delta g_{\mu\nu}}. \tag{142}$$

In the case of the conformal model, the r.h.s. of this equation simply vanishes, i.e., in purely conformal, Weyl-squared, gravity theory, the divergences of the effective action do not depend on α_i because the classical action satisfies the Noether identity for the conformal invariance. For the general model (30), the C^2, E_4 , and $\Box R$ terms in the action do not contribute to the r.h.s. of (142). Then, the gauge and parametrization dependencies are defined by the Einstein-Hilbert, cosmological, and R^2 terms. It is easy to get

对于共形模型，该方程的右侧简单等于零，也就是说，在纯共形、Weyl 平方引力理论中，有效作用量的发散不依赖于 α_i ，因为经典作用量满足共形不变性的诺特恒等式。对于一般模型 (30)，作用量中的 C^2, E_4 和 $\Box R$ 项对 (142) 的右侧没有贡献。此时规范和参数化依赖性由爱因斯坦-希尔伯特项、宇宙学项和 R^2 项决定。容易得到

$$\Gamma^{(1)}_{\text{div}}\left(\alpha_i\right) - \Gamma^{(1)}_{\text{div}}\left(\alpha^0_i\right) = f\left(\alpha_i\right) \int d^4x \sqrt{-g} \left\{ \frac{2\omega}{\lambda} \Box R - \frac{1}{\kappa^2}\left(R + 4\Lambda\right) \right\}. \tag{143}$$

The divergent coefficient of the $\Box R$ term depends on the gauge fixing; the same is true for the coefficients of the Einstein-Hilbert and cosmological terms. At the same time, there are two gauge-invariant combinations of these coefficients.

$\Box R$ 项的发散系数依赖于规范固定；爱因斯坦-希尔伯特项和宇宙学项的系数同样如此。同时，这些系数存在两个规范不变组合。

The easiest part is the gauge and parametrization dependence of the counterterms in superrenormalizable models with more than four derivatives, both polynomial or nonlocal. It is easy to see that these models do not have such dependencies. To get this result, we note that formula (139) is valid for all such models, both polynomial and nonlocal, with trading of $\varepsilon^{\mu\nu}_{(4)}$ to the variational derivative of the corresponding action $\varepsilon^{\mu\nu}$. As we have seen above, according to the power counting arguments, the divergences are given by local expressions with four, two, or zero derivatives of the metric. On the other hand, in all superrenormalizable models, the equations of motion $\varepsilon^{\mu\nu}$ have more than four derivatives of the metric. Thus, the nonzero r.h.s. of (139) is incompatible with the locality of the function $f_{\mu\nu}$, proving the statement about the universality of renormalization [20,44,84].

对于导数个数超过四的超可重整化模型 (无论是多项式还是非局域模型)，最容易分析其抵消项的规范和参数化依赖性。不难证明这类模型不存在这种依赖性。为得到这个结论，我们注意到公式 (139) 对所有这类模型 (多项式和非局域模型均成立) 都有效，只需要将 $\varepsilon^{\mu\nu}_{(4)}$ 替换为对应作用量 $\varepsilon^{\mu\nu}$ 的变分导数。正如上文所述，根据幂次计数论证，发散由含 4 个、2 个或 0 个度规导数的局部表达式给出。另一方面，在所有超可重整化模型中，运动方程 $\varepsilon^{\mu\nu}$ 含有超过四个度规导数。因此，(139) 非零的右侧与函数 $f_{\mu\nu}$ 的局域性不相容，这就证明了重整化普适性的结论 [20,44,84]。

One-Loop Divergences in Quantum GR

量子广义相对论中的单圈发散

The derivation of one-loop divergences in quantum GR has great historical [32] and practical importance. This calculation is a starting point for many other developments, in many different models, including pure QG, models of more and more complicated versions of pure QG, gravity coupled to quantum matter, etc. For these reasons, the review on perturbative QG should include this calculation and the list of the most important extensions and corresponding references.

量子广义相对论中单圈发散的推导具有重要的历史 [32] 与实践意义。该计算是诸多不同模型中许多后续发展的起点，这些模型包括纯量子引力、越来越复杂的纯量子引力版本、引力与量子物质耦合等。因此，关于微扰量子引力的综述理应包含这一计算，以及最重要的扩展工作列表和对应参考文献。

In the rest of this section, we repeat the original derivation of divergences in pure QG from the classical paper [32]. Since this is not complicated, we shall also include the nonzero cosmological constant, as it was done in [85]. The standard calculation uses the background field method based on (9), the heat-kernel expansion, and the Schwinger-DeWitt technique [1] (see also [7] for a detailed introduction).

在本节剩余部分，我们将重复经典论文 [32] 中对纯量子引力发散的原始推导。由于该过程并不复杂，我们还会像文献 [85] 那样纳入非零宇宙学常数。标准计算采用基于式 (9) 的背景场方法、热核展开以及施温格-德维特技术 [1](详细介绍参见文献 [7])。

The bilinear expansion of the action (22) is given in Eq. (48), and the gauge-fixing term with the two gauge-fixing parameters α and β is given by (25). The ghost action can be easily obtained from (27), but we postpone this part until fixing the values of α and β . For this, we rewrite (25) as

作用量 (22) 的双线性展开由式 (48) 给出，含两个规范固定参数 α 和 β 的规范固定项由式 (25) 给出。鬼作用量可很容易从式 (27) 得到，但我们将这部分推迟到确定 α 和 β 的值之后再处理。为此，我们将式 (25) 改写为

$$S_{gf} = \frac{1}{\alpha}\int d^4x\sqrt{-g}h^{\mu\upsilon}\left[g_{\mu\alpha}\nabla_\upsilon\nabla_\beta - \beta\left(g_{\mu\upsilon}\nabla_\alpha\nabla_\beta - g_{\alpha\beta}\nabla_\mu\nabla_\upsilon\right)\right.$$

$$\left.+\beta^2 g_{\mu\nu}g_{\alpha\beta}\Box\right]h^{\alpha\beta}\ . \tag{144}$$

Adding this expression to (48), we require that the sum includes the minimal operator $H_{\mu\upsilon,\alpha\beta}$,

将该表达式代入式 (48)，我们要求求和结果包含最小算子 $H_{\mu\upsilon,\alpha\beta}$,

$$S_{EH}^{(2)} + S_{gf} = \frac{1}{2}\int d^4x\sqrt{-g}h^{\mu\upsilon}H_{\mu\upsilon,\alpha\beta}h^{\alpha\beta},$$

$$H_{\mu\upsilon,\alpha\beta} = K_{\mu\upsilon,\alpha\beta}\Box + M_{\mu\upsilon,\alpha\beta}, \tag{145}$$

where $K_{\mu\upsilon,\alpha\beta}$ and $M_{\mu\upsilon,\alpha\beta}$ are c -number operators. This is achieved for $\alpha = 2$ and $\beta = 1/2$. After that, we arrive at the expression (145) with

其中 $K_{\mu\upsilon,\alpha\beta}$ 和 $M_{\mu\upsilon,\alpha\beta}$ 是 c 数算子，满足该要求的取值为 $\alpha = 2$ 和 $\beta = 1/2$ 。代入后我们得到表达式 (145)，其中

$$K_{\mu\upsilon,\alpha\beta} = \frac{1}{2}\left(\delta_{\mu\upsilon,\alpha\beta} - \frac{1}{2}g_{\mu\upsilon}g_{\alpha\beta}\right), \tag{146}$$

$$M_{\mu\upsilon,\alpha\beta} = R_{\mu\alpha\upsilon\beta} + g_{\upsilon\beta}R_{\mu\alpha} - \frac{1}{2}\left(g_{\mu\upsilon}R_{\alpha\beta} + g_{\alpha\beta}R_{\mu\upsilon}\right) - \frac{1}{2}R\left(\delta_{\mu\upsilon,\alpha\beta} - \frac{1}{2}g_{\alpha\beta}g_{\mu\upsilon}\right).$$

It is easy to see that the matrix $2K_{\rho\sigma,\alpha\beta}$ is equal to its own inverse,

不难看出矩阵 $2K_{\rho\sigma,\alpha\beta}$ 等于其自身的逆,

$$\left(\delta_{\mu\upsilon,\alpha\beta} - \frac{1}{2}g_{\mu\upsilon}g_{\alpha\beta}\right)\left(\delta^{\alpha\beta,\rho\sigma} - \frac{1}{2}g^{\alpha\beta}g^{\rho\sigma}\right) = \delta_{\mu\upsilon,}{}^{\rho\sigma}. \tag{147}$$

On the other hand, $\mathrm{Tr}\ln\left(2K_{\rho\sigma,\alpha\beta}\right)$ does not contribute to the divergences (e.g., in the dimensional regularization) since this operator has no derivatives. Thus, regarding the divergences,

另一方面，$\mathrm{Tr}\ln\left(2K_{\rho\sigma,\alpha\beta}\right)$ 对发散没有贡献 (例如维数正则化中)，因为该算子不含导数项。因此，针对发散而言,

$$\mathrm{Tr}\ln\left(H_{\rho\sigma,\alpha\beta}\right) = \mathrm{Tr}\ln\left(2K_{\mu\upsilon,}^{\rho\sigma}H_{\rho\sigma,\alpha\beta}\right) = \mathrm{Tr}\ln\left(H'_{\rho\sigma,\alpha\beta}\right)$$

$$= \mathrm{Tr}\ln\left(\delta_{\mu\nu,\alpha\beta}\Box + \Pi_{\rho\sigma,\alpha\beta}\right). \tag{148}$$

A small calculation gives

简单计算可得

$$\hat{\Pi} = \Pi_{\rho\sigma,\alpha\beta} = 2K_{\mu\nu,}{}^{\rho\sigma}M_{\rho\sigma,\alpha\beta} = M_{\mu\nu,\alpha\beta}. \tag{149}$$

It is evident that Eq. (148) enables one to use the standard Schwinger-DeWitt formula for the operator

显然式 (148) 允许我们对该算子使用标准的施温格-德维特公式

$$\hat{H} = \hat{1}\Box + 2\hat{h}^{\mu}\nabla_{\mu} + \hat{\Pi}. \tag{150}$$

For this, we need to define

为此，我们需要定义

$$\hat{S}_{\mu\nu} = \left[\nabla_{\nu}, \nabla_{\mu}\right]\hat{1} + \left(\nabla_{\nu}\hat{h}_{\mu} - \nabla_{\mu}\hat{h}_{\nu}\right) + \left(\hat{h}_{\nu}\hat{h}_{\mu} - \hat{h}_{\mu}\hat{h}_{\nu}\right) \tag{151}$$

and

以及

$$\hat{P} = \hat{\Pi} + \frac{\hat{1}}{6}R - \nabla_{\mu}\hat{h}^{\mu} - \hat{h}_{\mu}\hat{h}^{\mu}. \tag{152}$$

The divergent part of the one-loop effective action is an integral over the "magic" a_2 coefficient,

单圈有效作用量的发散部分是对“魔法”a_2 系数的积分，

$$\Gamma^{(1)}_{\mathrm{div}} = -\frac{\mu^{n-4}}{\varepsilon}\int d^n x\sqrt{-g}\,\mathrm{tr}\,\hat{a}_2 \mid, \tag{153}$$

where

其中

$$\hat{a}_2 \mid = \hat{a}_2(x,x) = \frac{\hat{1}}{180}\left(R^2_{\mu\nu\alpha\beta} - R^2_{\alpha\beta} + \Box R\right) + \frac{1}{2}\hat{P}^2 + \frac{1}{6}\left(\Box\hat{P}\right) + \frac{1}{12}\hat{S}^2_{\mu\nu}. \tag{154}$$

In our case, we have a simple situation because $\hat{h}_{\mu} = 0$. Thus,

在我们的情形中，由于 $\hat{h}_{\mu} = 0$ ，情况十分简单。因此，

$$\hat{P} = \hat{\Pi} + \frac{\hat{1}}{6}R \quad \text{and} \quad \hat{S}_{\mu\nu} = \left[\nabla_{\nu}, \nabla_{\mu}\right] \tag{155}$$

A simple calculation gives

简单计算可得

$$\widehat{P} = P_{\mu\upsilon,\alpha\beta} = \widehat{K}_1 + \widehat{K}_2 - \frac{1}{2}\widehat{K}_3 - \frac{5}{12}\widehat{K}_4 + \frac{1}{4}\widehat{K}_5, \quad \text{where}$$

$$\widehat{K}_1 = R_{\mu\alpha\upsilon\beta}, \ \widehat{K}_2 = g_{\upsilon\beta}R_{\mu\alpha}, \ \widehat{K}_3 = g_{\mu\upsilon}R_{\alpha\beta} + g_{\alpha\beta}R_{\mu\upsilon},$$

$$\widehat{K}_4 = \delta_{\mu\upsilon,\alpha\beta}R, \ \widehat{K}_5 = Rg_{\alpha\beta}g_{\mu\upsilon},$$

$$\text{and} \ \ \hat{S}_{\lambda\tau} = [S_{\lambda\tau}]_{\mu\upsilon,\alpha\beta} = -2R_{\mu\alpha\lambda\tau}g_{\upsilon\beta} \,. \tag{156}$$

For the sake of compactness, in all of these expressions, we assume automatic symmetrization over the couples of indices $(\mu\upsilon)$ and $(\alpha\beta)$. It is easy to get the following multiplication table:

为了简洁，所有这些表达式中，我们默认对指标对 $(\mu\upsilon)$ 和 $(\alpha\beta)$ 自动对称化，很容易得到如下乘法表:

$$\mathrm{tr}\,\widehat{K}_1 \cdot \widehat{K}_1 = \frac{1}{2}R^2_{\mu\nu\alpha\beta}, \mathrm{tr}\,\widehat{K}_2 \cdot \widehat{K}_2 = \frac{3}{2}R^2_{\mu\nu} + \frac{1}{4}R^2, \mathrm{tr}\,\widehat{K}_3 \cdot \widehat{K}_3 = 8R^2_{\mu\nu} + 2R^2,$$

$$\mathrm{tr}\,\widehat{K}_4 \cdot \widehat{K}_4 = 10R^2, \ \mathrm{tr}\,\widehat{K}_5 \cdot \widehat{K}_5 = 16R^2, \ \mathrm{tr}\,\widehat{K}_1 \cdot \widehat{K}_2 = -\frac{1}{2}R^2_{\mu\nu},$$

$$\mathrm{tr}\,\widehat{K}_1 \cdot \widehat{K}_3 = 2R^2_{\mu\nu}, \mathrm{tr}\,\widehat{K}_1 \cdot \widehat{K}_4 = -\frac{1}{2}R^2, \mathrm{tr}\,\widehat{K}_1 \cdot \widehat{K}_5 = R^2, \mathrm{tr}\,\widehat{K}_2 \cdot \widehat{K}_3 = 2R^2_{\mu\nu},$$

$$\mathrm{tr}\,\widehat{K}_2 \cdot \widehat{K}_4 = \frac{5}{2}R^2, \ \mathrm{tr}\,\widehat{K}_2 \cdot \widehat{K}_5 = R^2, \ \mathrm{tr}\,\widehat{K}_3 \cdot \widehat{K}_4 = 2R^2, \mathrm{tr}\,\widehat{K}_3 \cdot \widehat{K}_5 = 8R^2,$$

$$\mathrm{tr}\,\widehat{K}_4 \cdot \widehat{K}_5 = 4R^2, \ \ \text{and} \ \ \hat{S}_{\lambda\tau} \cdot \hat{S}_{\lambda\tau} = -6R^2_{\mu\nu\alpha\beta}. \tag{157}$$

Substituting these values into Eq. (154), we get

将这些值代入式 (154)，我们得到

$$\mathrm{tr}\left(\frac{1}{2}\widehat{P} \cdot \widehat{P} + \frac{1}{12}\hat{S}_{\lambda\tau} \cdot \hat{S}^{\lambda\tau}\right) = R^2_{\mu\nu\alpha\beta} - 3R^2_{\mu\nu\alpha\beta} + \frac{59}{36}R^2 + \frac{26}{3}R\Lambda + 20\Lambda^2.$$

(158)

In this and subsequent expressions, we ignore the total derivative term $\Box R$. The complete expressions can be found, e.g., in the original publication [83], including for the general parametrization of the quantum metric (132).

在本式及后续表达式中，我们忽略全导数项 $\Box R$ 。完整表达式可参见原始文献 [83]，其中包含量子度规 (132) 的一般参数化形式。

For the ghost action, we obtain

对于鬼场作用量，我们得到

$$\frac{\delta\chi^{\mu}}{\delta h_{\rho\sigma}} = \delta^{\mu\lambda,\rho\sigma}\nabla_{\lambda} - \frac{1}{2}g^{\rho\sigma}\nabla^{\mu},\ R^{\upsilon}_{\rho\sigma,} = -\delta^{\upsilon}_{\rho}\nabla_{\sigma} - \delta^{\upsilon}_{\sigma}\nabla_{\rho}.$$

Then

于是

$$M^{\upsilon}_{\mu} = \frac{\delta\chi^{\mu}}{\delta h_{\rho\sigma}}R^{\upsilon}_{\rho\sigma,} = -\left(\delta^{\upsilon}_{\mu}\Box + R^{\upsilon}_{\mu}\right). \tag{159}$$

This is a minimal vector operator that enables one to use the standard Schwinger-DeWitt formula (153). For the commutator, we get

这是一个极小矢量算符，可使用标准的施温格-德维特公式 (153)。对于对易子，我们得到

$$\hat{S}_{\lambda\tau} = \left[\hat{S}_{\lambda\tau}\right]_{\alpha,\beta} = -R_{\alpha\beta\lambda\tau}. \tag{160}$$

Thus,

因此，

$$\mathrm{tr}\left(\frac{1}{2}\hat{P}\cdot\hat{P} + \frac{1}{12}\hat{S}_{\lambda\tau}\cdot\hat{S}^{\lambda\tau}\right)_{\text{ghost}} = -\frac{1}{12}R^{2}_{\mu\nu\alpha\beta} + \frac{3}{2}R^{2}_{\mu\nu\alpha\beta} + \frac{2}{9}R^{2}. \tag{161}$$

Finally, replacing all the expressions above into (154) and taking into account that, in the $h_{\mu\nu}$ sector, $\mathrm{tr}\,\hat{1} = \delta_{\mu\nu\alpha\beta}\delta^{\mu\nu\alpha\beta} = 10$ and, in the ghost sector, $\mathrm{tr}\,\hat{1}_{\text{ghost}} = 4$, we arrive at the famous result of [32],

最后,将上述所有表达式代入 (154),并考虑到在 $h_{\mu\nu}$ 区 $\mathrm{tr}\,\hat{1} = \delta_{\mu\nu\alpha\beta}\delta^{\mu\nu\alpha\beta} = 10$,在鬼场区 $\mathrm{tr}\,\hat{1}_{\text{ghost}} = 4$ ，我们得到文献 [32] 中著名的结果，

$$\Gamma^{1,\,\text{total}}_{\text{div}} = \frac{i}{2}\,\mathrm{Tr}\,\mathrm{Ln}\,\hat{H} - i\,\mathrm{Tr}\,\mathrm{Ln}\,\hat{H}_{\text{ghost}} \tag{162}$$

$$= -\frac{\mu^{n-4}}{\varepsilon}\int d^{n}x\sqrt{-g}\left\{\frac{53}{45}R^{2}_{\mu\nu\alpha\beta} - \frac{361}{90}R^{2}_{\alpha\beta} + \frac{43}{36}R^{2} + \frac{26}{3}R\Lambda + 20\Lambda^{2}\right\}$$

$$= -\frac{\mu^{n-4}}{\varepsilon}\int d^{n}x\sqrt{-g}\left\{\frac{53}{45}E_{4} + \frac{7}{10}R^{2}_{\alpha\beta} + \frac{1}{60}R^{2} + \frac{26}{3}R\Lambda + 20\Lambda^{2}\right\},$$

where, as usual, $\varepsilon = (4\pi)^{2}(n-4)$, and μ is the renormalization parameter.

其中，按惯例 $\varepsilon = (4\pi)^{2}(n-4)$ ，μ 是重整化参数。

As we already know from section "Gauge-Fixing Dependence in Quantum GR," the coefficient $\frac{53}{45}$ of the Gauss-Bonnet term is invariant, while other coefficients can be modified by changing the gauge-fixing conditions or the parametrization of the quantum metric. The second invariant coefficient in (137) can be easily found by using the on-shell conditions $R_{\mu\nu} = -\Lambda g_{\mu\nu}$ and $R = -4\Lambda$. The result of this operation is

正如我们从“量子广义相对论的规范固定依赖性”一节已经了解到的, 高斯-博内项的系数 $\frac{53}{45}$ 是不变量，而其他系数可以通过改变规范固定条件或量子度规参数化来修改。利用在壳条件 $R_{\mu\nu} = -\Lambda g_{\mu\nu}$ 和 $R = -4\Lambda$ 可以很容易得到 (137) 中的第二个不变系数，最终结果为

$$c_{\,\mathrm{inv}} = -\frac{58}{5}\Lambda^2. \tag{163}$$

This means, the invariant, on-shell, version of (162) is

这意味着，(162) 的不变在壳形式为

$$\Gamma^{1,\,\mathrm{total}}_{\mathrm{div}} == -\frac{\mu^{n-4}}{\varepsilon}\int d^nx\sqrt{-g}\Big\{\frac{53}{45}E_4 - \frac{58}{5}\Lambda^2\Big\}. \tag{164}$$

To conclude this section, let us make a few observations.

在本节最后，我们做几点说明。

(i) The reader could note that in the formulas in the previous section, e.g., in (143), we used $\int d^4x\sqrt{-g}$, while in this section, the more complicated integration rule $\int d^nx\mu^{n-4}\sqrt{-g}$ was used. The point is that the two formulas are equivalent when it concerns the divergences. The expression with $\int d^4x$ is just shorter and the one with $\int d^nx$ may be more useful, e.g., for deriving the beta function in the minimal subtraction scheme of renormalization.

(i) 读者可能已经注意到，在上一节的公式 (例如 (143)) 中，我们使用了 $\int d^4x\sqrt{-g}$ ，而在本节中使用了更复杂的积分规则 $\int d^nx\mu^{n-4}\sqrt{-g}$ 。事实上，二者在讨论发散问题上是等价的: 带 $\int d^4x$ 的表达式更简洁，而带 $\int d^nx$ 的表达式更有用，例如可用于推导极小减除重整化方案中的 β 函数。

(ii) One may be curious why we identify the square of the Riemann tensor in the formula for the invariant part of divergences in Eq. (138) with the Gauss-Bonnet topological invariant E_4 and not with the square of the Weyl tensor. This is an important question and we shall give an extensive answer to it.

(ii) 读者可能会好奇，为什么我们将 (138) 式发散不变部分公式中的黎曼张量平方等同于高斯-博内拓扑不变量 E_4 ，而不是外尔张量平方。这是一个重要问题，我们将给出详尽解答。

This issue requires addressing the nonlocal finite contributions corresponding to the logarithmic UV divergences. One can calculate these nonlocal terms directly (see, e.g., [77, 86] for the heat-kernel approach that can be used for this purpose), or using Feynman diagrams, including in the theory of massive fields [87] (see also [7, 88, 89]). In the massless limit, the nonlocal logarithmic terms correspond the higher-derivative divergences (134) and have the form [7]

这个问题需要处理对应对数紫外发散的非局域有限贡献。可以直接计算这些非局域项 (例如，参见 [77, 86] 中可用于此目的的热核方法)，也可以使用费曼图计算，包括在有质量场理论中 [87](另见 [7, 88, 89])。在无质量极限下，非局域对数项对应高阶导数发散 (134)，形式为 [7]

$$\Gamma^{(1)}_{\text{fin},\,HD} = \int d^4x\sqrt{-g}\Big\{c_1 R_{\mu\nu\alpha\beta}\ln\Big(-\frac{\Box}{\mu^2}\Big)R^{\mu\nu\alpha\beta}$$

$$+c_2 R_{\alpha\beta}\ln\Big(-\frac{\Box}{\mu^2}\Big)R^{\alpha\beta} + c_3 R\ln\Big(-\frac{\Box}{\mu^2}\Big)R\Big\}. \tag{165}$$

Let us note that one can also derive the nonlocal form factor for the Einstein-Hilbert term [88,89], regardless there is an ambiguity even in the nonlocal representation of the proper action of GR [90]. However, what we need is just the expression (165), which is the physical reflection of the divergences.

需要指出，即便广义相对论正规作用量的非局域表示本身存在歧义 [90]，我们仍然可以推导出爱因斯坦-希尔伯特项的非局域形状因子 [88,89]。但我们所需要的只是式 (165)，它是发散的物理体现。

The correspondence between divergences and nonlocal terms holds independent on the choice of gauge fixing and parametrization, such that the transformations (136) apply to the coefficients of the logarithmic terms in (165). As we know from section "Propagator of Metric and the Barnes-Rivers Projectors," all three parts of this expression affect the propagation of the spin-2 or spin-0 modes of the metric. The ambiguity (136) affects this propagation and, in particular, can eliminate all of it. As we know from the discussion of propagator in the general models (39) with an extra term (40), the unique part of the expression which does not affect the propagation of the spin-2 or spin-0 modes of the metric is the generalized Gauss-Bonnet term

发散与非定域项之间的对应关系与规范固定和参数化的选择无关，因此变换 (136) 适用于 (165) 中对数项的系数。正如我们从"度规传播子与巴恩斯-里弗斯投影算符"一节所知，该表达式的三个部分都会影响度规自旋-2 模或自旋-0 模的传播。歧义 (136) 会影响该传播，尤其还能完全消除这种传播。正如我们对带有额外项 (40) 的一般模型 (39) 的传播子讨论所知，该表达式中不影响度规自旋-2 模或自旋-0 模传播的唯一部分是广义高斯-博内项

$$\Gamma^{(1)}_{GB,\,\text{nonloc}} = -c_1\int d^4x\sqrt{-g}\Big\{R_{\mu\nu\alpha\beta}\ln\Big(-\frac{\Box}{\mu^2}\Big)R^{\mu\nu\alpha\beta}$$

$$-4R_{\alpha\beta}\ln\Big(-\frac{\Box}{\mu^2}\Big)R^{\alpha\beta} + R\ln\Big(-\frac{\Box}{\mu^2}\Big)R\Big\}. \tag{166}$$

This means, the invariant coefficient c_1 in (137) should be attributed to the E_4 and not to the Weyl-squared term. In case of QG based on GR, this coefficient is 53/45. There may be further contributions from matter fields, which do not depend on the gauge fixing in the QG sector of the theory.

这意味着，(137) 中的不变系数 c_1 应当归属于 E_4，而非外尔平方项。在基于广义相对论的量子引力中，该系数为 53/45。物质场还可能带来额外贡献，这类贡献不依赖于该理论量子引力部分的规范固定选择。

(iii) As one can see already from the power counting arguments, the renormalization of the the Einstein-Hilbert term in QG based on GR is possible only owing to the nonzero cosmological constant Λ. This feature holds beyond the one-loop order and can be seen as a general result. However, this statement corresponds to the most universal, logarithmic, divergences. The quadratic divergences of the Einstein-Hilbert type are possible even for $\Lambda = 0$. However, we know that these divergences depend on the choice of regularization and, therefore, the physical results which one can obtain from quadratic divergences always have uncertain physical sense. This issue was discussed in more detail, e.g., in [7].

(iii) 仅从幂次数计数论证就可以看出，在基于广义相对论的量子引力中，爱因斯坦-希尔伯特项能够重整化，完全依赖于非零宇宙学常数 Λ 。这一性质在单圈阶之外依然成立，属于一般性结论。不过该结论针对的是最普适的对数发散。即使对于 $\Lambda = 0$ ，爱因斯坦-希尔伯特型的二次发散也有可能存在。但我们知道，这类发散依赖于正则化方案的选择，因此从二次发散得到的物理结果其物理意义始终不明确。例如，文献 [7] 已经对该问题做过更详细的讨论。

On-Shell Renormalization Group in Quantum GR

量子广义相对论中的在壳重整化群

The formulation of the renormalization group running in QG is a complicated problem because renormalizable and superrenormalizable models of QG always have massive ghosts and other massive degrees of freedom, such that the physical sense of the running (and quantum effects, in general) in these models is not clear. The reason is that loop contributions in these models come from the functional integral over the massless mode of the gravitational field and, also, over the massive modes. It is a standard assumption that the contributions of massive fields vanish at the energy scale below their masses. In the fourth-derivative QG, this expectation was formulated, for the first time, in [24]. Nowadays, it is partially supported by the calculations in semiclassical theories [87,89] and in the toy model of QG [91]. On the other hand, in QG all massive modes have masses of the Planck order of magnitude (see the discussion of this issue in [37]). Thus, the physical applicability of the renormalization group running in fourth- and higher-order models of QG is restricted by the UV domain, with energies above the Planck scale.

量子引力中重整化群跑动的构造是一个复杂问题，因为可重整和超可重整的量子引力模型总是存在大质量鬼场与其他大质量自由度，导致这些模型中跑动 (以及广义上的量子效应) 的物理意义并不明确。原因在于这些模型的圈贡献既来自引力场无质量模的泛函积分，也来自大质量模的泛函积分。标准假设认为，大质量场的贡献在能量低于其质量时会消失。在四阶导数量子引力中，这一猜想最早由文献 [24] 提出。如今，它得到了半经典理论计算 [87,89] 以及量子引力玩具模型计算 [91] 的部分支持。另一方面，量子引力中所有大质量模的质量都处于普朗克量级 (相关讨论见文献 [37])。因此，重整化群跑动在四阶及高阶量子引力模型中的物理适用性被限制在能量高于普朗克能标的紫外区域。

Thus, despite the running of parameters of the actions in all renormalizable QG models that might have great importance from the theoretical perspective, its application to any kind of physics is unclear. Since the universal QG theory in the IR is quantum GR [92], we can try to explore the running using this model. However, the use of the renormalization group to explore the running in the quantum GR is a nontrivial issue because the theory is not renormalizable. Assuming that all massive degrees of freedom decouple in the IR (below the Planck scale) and the physically interesting running are in the region below the Planck scale of energies, we arrive at the subject of effective approach to QG, which is treated in a separate section of this handbook.

因此，尽管从理论角度看，所有可重整量子引力模型中的作用量参数跑动可能十分重要，但它对任何物理研究的应用都尚不明确。由于红外区域的普适量子引力理论是量子广义相对论 [92]，我们可以尝试在该模型下探究跑动。然而，在量子广义相对论中使用重整化群探究跑动是一个非平凡问题，因为该理论不可重整。假设所有大质量自由度都在红外区域 (普朗克能标以下) 退耦，且物理上有意义的跑动发生在能量低于普朗克能标的区域，我们就进入了量子引力有效方法的研究范畴，本手册将在单独章节讨论这一内容。

Let us consider a version of the renormalization group running in QG which enables one to avoid the mentioned difficulties. This version of the renormalization group is not perfectly well defined, but it gives an idea of what we can expect from the running. The on-shell version of the renormalization group uses the expressions (138) for the classical action and one-loop divergences on the classical equations of motion. The on-shell divergences are universal, i.e., do not depend on the parametrization of the quantum metric and on the gauge-fixing choice. Ignoring the topological term $c_1 E_4$, the aforementioned expressions can be used to perform the renormalization on shell and, consequently, to achieve the renormalization group running.

下面我们讨论量子引力中一种可以规避上述困难的重整化群跑动方案。这种重整化群方案的定义并不完美，但它可以给出我们对跑动的预期。在壳版本的重整化群利用了经典作用量的表达式 (138)，以及经典运动方程上的单圈发散。在壳发散是普适的，即不依赖于量子度规的参数化方式也不依赖于规范固定选择。忽略拓扑项 $c_1 E_4$ ，我们可以利用上述表达式完成在壳重整化，进而得到重整化群跑动。

Let us define the dimensionless parameter

我们来定义一个无量纲参数

$$\gamma = \kappa^2 \Lambda \tag{167}$$

In terms of this parameter, the classical action and the one-loop counterterm (both on shell) have the form

用该参数表示时，经典作用量和单圈抵消项 (均为在壳形式) 可写为

$$\begin{aligned} S_{EH}\big|_{\text{on-shell}} &= -\frac{1}{\kappa^4} \int d^n x \sqrt{-g} \mu^{n-4} (-2\gamma), \\ \Delta S^{(1)}\big|_{\text{on-shell}} &= \frac{1}{\varepsilon} \cdot \frac{1}{\kappa^4} \int d^n x \sqrt{-g} \mu^{n-4} \left(-\frac{58}{5}\gamma^2\right). \end{aligned} \tag{168}$$

As these two expressions have an identical dependence of the metric, we can remove the divergence by making the renormalization transformation,

由于这两个表达式对度规的依赖关系完全一致，我们可以通过做重整化变换消除发散，

$$\gamma_0 = \mu^{n-4} \left(\gamma - \frac{29}{5\varepsilon}\gamma^2\right), \tag{169}$$

from what immediately follows the general β -function in n spacetime dimensions,

由此可直接得到 n 维时空下的广义 β 函数,

$$\mu \frac{d\gamma}{d\mu} = -(n-4)\gamma - \frac{29}{5(4\pi)^2}\gamma^2. \tag{170}$$

The standard β -function for γ can be obtained in the limit $n \to 4$ and we arrive at the renormalization group equation that corresponds to an asymptotically free theory,

在 $n \to 4$ 极限下可以得到针对 γ 的标准 β 函数，我们最终得到对应渐近自由理论的重整化群方程,

$$\mu \frac{d\gamma}{d\mu} = \beta_\gamma = -a^2\gamma^2, \; a^2 = \frac{29}{5(4\pi)^2}. \tag{171}$$

The solution of this equation can be easily found if we set the initial value at some scale, $\gamma(\mu_0) = \gamma_0$,

如果给定某一能标下的初值 $\gamma(\mu_0) = \gamma_0$ ，就可以很容易得到该方程的解,

$$\gamma(\mu) = \frac{\gamma_0}{1 + a^2\gamma_0 \ln(\mu/\mu_0)}. \tag{172}$$

The physical interpretation of this solution meets several difficulties, so let us present some points discussing this subject.

该解的物理解释存在若干困难，因此我们介绍一些与该问题相关的讨论要点。

1. It is possible to identify μ with one or another physical parameter, in different physical situations. As far as γ is the dimensionless cosmological constant, the standard application of the solution should be in cosmology and then the natural choice of the scale is the Hubble parameter [93-95].

1. 在不同物理场景中，我们可以将 μ 对应为不同的物理参数。由于 γ 是无量纲宇宙学常数，该解的标准应用场景是宇宙学，此时哈勃参数是能标的自然选择 [93-95]。

2. The cosmological constant that enters the definition (167) is not the one which is responsible for the observed accelerated expansion of the Universe. The observable density of the cosmological constant $\rho_\Lambda^{obs} = \Lambda/(8\pi G)$ is a sum of the vacuum quantity $\rho_\Lambda^{\mathrm{vac}} = \rho_\Lambda^{\mathrm{vac}}$ and the induced density $\rho_\Lambda^{\mathrm{ind}}$ coming from, e.g., the electroweak phase transition. Both summands are many orders of magnitude greater than the observed quantity $\rho_\Lambda^{\mathrm{obs}}$. For this reason, the numerical value of γ_0 may be small, but it is not that small as some reader might think. This issue was discussed in detail in [96], with the consideration based on the unique effective action formalism [97-99]. This interesting approach is discussed in a separate chapter of this section and does not fit the present review. However, we shall briefly discuss the differences between the two universal equations for $\gamma(\mu)$ below.

2. 进入定义式 (167) 的宇宙常数并非是导致观测到的宇宙加速膨胀的那个宇宙常数。可观测的宇宙常数密度 $\rho_\Lambda^{obs} = \Lambda/(8\pi G)$ 是真空项 $\rho_\Lambda^{\rm vac} = \rho_\Lambda^{\rm vac}$ 与例如电弱相变诱发的感应密度 $\rho_\Lambda^{\rm ind}$ 之和。这两项都比观测值 $\rho_\Lambda^{\rm obs}$ 大很多个数量级。因此，γ_0 的数值可以很小，但并没有小到部分读者可能认为的程度。该问题已在文献 [96] 中结合唯一有效作用量形式 [97-99] 做了详细讨论。这种有趣的方法会在本节单独一章展开讨论，不在本综述的讨论范围内。不过我们会在下文简要讨论针对 $\gamma(\mu)$ 的两个普适方程之间的差异。

3. Another interesting point is that if γ_0 is sufficiently small or negligible, this equation can be regarded as non-perturbative. To see this, it is sufficient to pick up the power counting in the quantum GR, as it was presented above.

3. 另一个有趣的点在于，如果 γ_0 足够小或可以忽略，该方程可被视为非微扰方程。要理解这一点，只需像上文那样对量子广义相对论进行幂次计数即可。

To end this section, let us compare the running (172) with the one in the effective approach to quantum GR with cosmological constant [96] (I am grateful to Breno Giacchini and Tibério de Paula Netto for the stimulating discussions of this issue.) (see also earlier work [99] with the same result but without effective interpretation). In the effective approach, the higher-derivative terms should be neglected, but without these terms, the invariants (137) cannot be constructed. The way out is to use the Vilkovisky-DeWitt scheme of unique effective action, which does not depend on parametrization or gauge fixing by construction [97,98]. The calculations in this case give [99]

在本节的最后，我们将 (172) 给出的跑动行为与带宇宙常数的量子广义相对论有效方案中的跑动行为 [96] 做对比 (感谢 Breno Giacchini 与 Tibério de Paula Netto 就该问题发起的启发性讨论)(另可参考得出了相同结论但未给出有效诠释的早期工作 [99])。在该有效方案中，高阶导数项应当被忽略，但丢掉这些项之后就无法构造不变量 (137)。解决方法是使用维尔科维斯基-德维特唯一有效作用量方案，该方案在构造上就不依赖于参数化与规范固定 [97,98]。这种情形下的计算给出 [99]

$$\Gamma^{1,\,\rm total}_{\rm div} = -\frac{\mu^{n-4}}{\varepsilon}\int d^n x\sqrt{-g}\Big\{\frac{53}{45}E_4 + \frac{121}{60}R^2_{\alpha\beta} - \frac{29}{60}R^2 + 8R\Lambda + 12\Lambda^2\Big\}. \tag{173}$$

This is different from (162), but it is easy to check that the on-shell expression (164) is the same. But in this case, we do not need to rely on the on-shell version. Neglecting all fourth-derivative terms in (173) and using the standard formalism, one gets the renormalization group equations for the constants in the action (22),

这与 (162) 式不同，但很容易验证它和壳表达式 (164) 是一致的。不过在这种情形下我们不需要依赖于 on-shell 版本。忽略 (173) 式中所有四阶导数项并使用标准形式，我们就能得到作用量 (22) 中各常数的重整化群方程

$$\mu\frac{\rm d}{{\rm d}\mu}\frac{1}{\kappa^2} = \frac{8\Lambda}{(4\pi)^2}, \tag{174}$$

$$\mu\frac{{\rm d}\Lambda}{{\rm d}\mu} = -\frac{2\Lambda^2\kappa^2}{(4\pi)^2}. \tag{175}$$

In this case, the equation for the dimensionless combination (167) has the same general form (171), but this time with the coefficient

这种情形下，无量纲组合 (167) 满足的方程仍具有相同的一般形式 (171)，只是系数变为

$$a^2 = \frac{10}{(4\pi)^2}. \tag{176}$$

This change of the beta function illustrates the role of the physical interpretation in QG. The difference between the two cases is that in (171) we do not ignore the terms with four derivatives and get the invariant beta function by using the on-shell condition. On the contrary, in (176) we follow the effective approach, ignore higher-derivative terms, and use Vilkovisky's unique effective action to provide an invariant result.

贝塔函数的这种变化体现了量子引力中物理解释的作用。两种情形的差异在于: 在 (171) 式中我们没有忽略四阶导数项，并且利用 on-shell 条件得到了不变贝塔函数；与之相反，在 (176) 式中我们遵循有效方案，忽略了高阶导数项，并使用维尔科维斯基唯一有效作用量得到了不变结果。

Both approaches look satisfactory and both are not perfect. Which one is better? In a "normal" physical model the answer would be probably given by experiment, but in QG this is not an option.

两种方法看起来都符合要求，但也都不够完美。哪一种更好呢？在“常规”物理模型中，答案大概率会由实验给出，但量子引力无法做到这一点。

One-Loop Divergences in Other Models of QG

其他量子引力模型中的单圈发散

There is a vast number of publications on the one-loop calculations and we do not pretend to make a complete review or even list the most relevant of these works (including the ones of the present author) here. In what follows, I will separate those papers which were first of the kind for the most relevant models.

关于单圈计算的出版物数量繁多，本文并不打算对此类工作 (包括本文作者本人的工作) 做完整综述，甚至都无法列出其中最相关的部分。下文我将仅针对最重要模型中的开创性论文进行区分介绍。

The first of these works concerned the one-loop divergences in quantum GR coupled to quantum matter. Already in the first paper with one-loop calculations, 't Hooft and Veltman derived also the divergences for QG coupled to the minimal scalar field [32]. It is interesting that the case of quantum GR coupled to the nonminimal quantum scalar ϕ (nonminimal means the presence of the term $\xi R\phi^2$) can be reduced to the calculations with a minimal scalar by the change of variables, i.e., the conformal transformation [100]. However, the correspondence with the direct calculation [101] required much greater effort and was achieved only two decades later in [102]. This example shows importance and complications related to the choice of parametrization. The calculations in gravity-vector (Abelian and nonabelian) and gravity-fermion models were done almost at the same time as [32] by Deser, van Nieuwenhuisen et al. in [33]. Let us note that more general combinations of quantum fields including QG were explored in supergravity, but since there is a special section of this handbook about supergravity, we do not need to discuss this part here. The important general result of the mentioned calculations is that, for matter-gravity systems, the one-loop divergences do not vanish on shell [32, 33].

这类工作最早研究的是耦合量子物质的量子引力中的单圈发散。早在第一篇单圈计算论文中，'t Hooft 和 Veltman 就推导出了耦合最小标量场的量子引力的发散 [32]。有趣的是，耦合非最小量子标量 ϕ 的量子引力 (非最小指存在项 $\xi R\phi^2$) 可以通过变量替换也就是共形变换约化为最小标量的计算 [100]。但它和直接计算 [101] 的对应结果却花费了极大精力，直到二十年后才在文献 [102] 中完成。这个例子体现了参数化选择的重要性与复杂性。Deser、van Nieuwenhuisen 等人在 [33] 中几乎和 [32] 同时完成了引力-矢量 (阿贝尔和非阿贝尔) 模型以及引力-费米子模型的计算。值得一提的是，超引力中已经研究了包含量子引力的更一般场组合，但由于本手册有专门章节介绍超引力，我们在此无需讨论这部分内容。上述计算得到的重要一般性结论是: 对于物质-引力系统，单圈发散在质壳上并不为零 [32, 33]。

The one-loop divergences were calculated in other models of QG, including fourth-derivative QG with the action (30) and the polynomial superrenormalizable model (34). The case of the theory (30) is treated in full detail in another chapter of this section of our handbook and we will not discuss the technical details here. From the general perspective, this calculation was relevant for several reasons. There is a growing understanding that the construction of QG, which is not restricted to the low-energy domain, cannot be successful without a higher derivative and the same is true in the semiclassical theory. Then, one of the ”traditional” expectations is to advance in the problem of ghosts and instabilities by exploring quantum corrections and, at the first place, the logarithmic contributions [59,60]. As we already know, these contributions are directly related to the UV divergences

单圈发散已经在其他量子引力模型中完成计算，包括作用量为 (30) 的四阶导数量子引力和多项式超可重整模型 (34)。本手册本部分的另一章节已经详细讨论了理论 (30) 的情形，因此我们在此不讨论技术细节。从一般性角度看，该计算之所以重要有多个原因。目前越来越多的认识表明，不局限于低能区域的量子引力构造离不开高阶导数，半经典理论也是如此。其次，“传统”期待之一是通过研究量子修正，首先是对数贡献，推进鬼和不稳定性问题的研究 [59,60]。正如我们已经知道的，这些贡献和紫外发散直接相关

Historically, the calculation of divergences in the fourth-derivative QG started in the work [103], using Feynman diagrams and separating massless and massive internal lines. This way to make calculations was never used after that (up to our knowledge) as all subsequent calculations were performed by technically a more simple heat-kernel (or Schwinger-DeWitt) technique, adapted for the four-derivative operators in [24] and later on, in a more systematic way, in [25]. However, the diagrammatic approach of the pioneering work of Julve and Tonin [103] may be, by the end of the day, rather important. The reason is that using diagrams enable one to separate the graphs with only massless internal lines from the ones which have the lines of the massive degrees of freedom (i.e., ghosts, in the present case). This method can be used to explore the IR limit of the quantum corrections in higher-derivative theories. As we already mentioned above, in this area we have only the general statement about the universality of quantum GR as an effective theory of QG in the IR [24,92] and the calculations of decoupling in the models of massive matter fields [87,91] (see further references therein) that can be used as toy models for the problem of decoupling massive degrees of freedom in QG. Let us stress that completing the program of decoupling in the higher-derivative models would be important for improved understanding of QG as a whole.

从历史上看，四阶导数量子引力中的发散计算最早始于文献 [103]，该文使用费曼图，分离了无质量和有质量内线。据我们所知，这种计算方法此后再也没有被使用过，因为后续所有计算都采用了技术上更简单的热核 (即 Schwinger-DeWitt) 技术，该技术在文献 [24] 中被适配到四导数算子，之后文献 [25] 对其进行了更系统的推广。然而，这项由 Julve 和 Tonin 完成的开创性工作所采用的图论方法，说到底仍然相当重要。原因在于，使用费曼图可以将仅含无质量内线的图和含大质量自由度 (在当前情形下就是鬼) 内线的图分离开来。该方法可以用来研究高阶导数理论中量子修正的红外极限。正如我们上文提到的，在该领域我们目前只有一般性结论: 量子引力作为量子引力的有效理论在红外具有普适性 [24,92]，以及大质量物质场模型中退耦的计算结果 [87,91](更多相关引用参见该文献)，这些结果可以作为量子引力中大质量自由度退耦问题的玩具模型。需要强调的是，完成高阶导数模型中的退耦方案对于全面增进对量子引力的理解十分重要。

The next important step was done in the seminal paper [24] by Fradkin and Tseytlin. This was the first publication where the relevance of the weight operator (32) for quantum contributions was noted; there was also the first heat-kernel treatment of the fourth-derivative operator and of the nonminimal vector operator, the first discussion of the IR decoupling in higher- derivative theories, and the first observation concerning the difference between general and conformal QG models. We can say that [24] paved the way for the development of a large part of QG in the subsequent decades. Furthermore, the correct result for the general and conformal versions of the fourth-derivative QG models was obtained in [104] and [105], respectively. Finally, there was a confirmation of the correctness of these results in [43, 106], where another interesting aspect of QG was addressed. Much earlier, in the paper [107], it was noted that the Gauss-Bonnet term $\int E_4$ may play a nontrivial role in QG. The point is that the topological nature of this term becomes badly seen in the perturbative approach, especially if one uses the dimensional regularization. In the previous sections we saw that the topological term does not contribute to the propagator, in any dimension n . However, for $n \neq 4$ it does affect the vertices. The question posed in [107] was whether it is true that the contributions of these vertices cancel in the divergent part of effective action, or even in the finite part of it. The answer is that the divergences really do not depend on the $\int E_4$ term, and this dependence takes place only in the local finite terms.

下一个重要步骤由弗拉德金 (Fradkin) 和采伊特林 (Tseytlin) 在开创性论文 [24] 中完成。这是第一篇指出权重算符 (32) 对量子贡献的相关性的出版物；文中还首次对四阶导数算符和非最小矢量算符进行了热核处理，首次讨论了高阶导数理论中的红外退耦，并且首次观测到广义量子引力模型和共形量子引力模型之间存在差异。可以说，[24] 为后续几十年量子引力大部分领域的发展铺平了道路。此外，四阶导数量子引力模型的广义版本和共形版本的正确结果分别在 [104] 和 [105] 中得到。最终，[43, 106] 证实了这些结果的正确性，这些文献还探讨了量子引力的另一个有趣方面。更早的时候，论文 [107] 就指出，高斯-博内项 $\int E_4$ 可能在量子引力中发挥非平凡作用。关键在于，该项的拓扑性质在微扰方法中很难显现，尤其是使用维数正规化的时候。在前几节中我们已经看到，拓扑项在任意维数 n 下都不对传播子有贡献。然而，对于 $n \neq 4$ ，它确实会对顶点产生影响。[107] 提出的问题是: 这些顶点的贡献是否真的会在有效作用量的发散部分 (甚至有限部分) 抵消。答案是: 发散确实与 $\int E_4$ 项无关，这种依赖仅出现在局部有限项中。

Finally, let us mention the derivation of the one-loop divergences in the polynomial, superrenormalizable models of QG [84]. From the technical side, these calculations require the generalized Schwinger-DeWitt technique developed by Barvinsky and Vilkovisky [25].

最后，我们来提及多项式超可重整化量子引力模型中一圈发散的推导 [84]。从技术层面看，这些计算需要用到巴尔温斯基 (Barvinsky) 和维尔科夫斯基 (Vilkovisky) 发展的广义施温格-德维特技术 [25]。

Concluding Discussion

结论讨论

Quantum gravity is intended as a theory to be most relevant in the vicinity of the singularities, i.e., at extremely high energies. The idea to formulate QG in a way of perturbative QFT, which proved efficient in other theories, is the first that comes to mind when one decides how to quantize gravity. In this sense, the perturbative approach is the most basic part of the whole QG program.

量子引力是一套原本预期在奇点附近、也就是极高能区域发挥核心作用的理论。当人们探索引力的量子化方案时，借助经其他理论验证有效的微扰量子场论框架来构造量子引力，是最先浮现的思路。从这个角度来说，微扰方案是整个量子引力研究中最基础的部分。

To a great extent, the current situation in perturbative QG is similar to the one in QG, in general. We have many approaches to incorporating the quantum effects of gravity within the well-established perturbative formalism of quantum field theory. On the other hand, there is no perfect model of quantum gravity. The theory based on GR is a promising candidate to describe low-energy effects (see the corresponding section on effective QG), but it has serious problems with non-renormalizability in the UV. From another side, the fourth- derivative model is renormalizable and enables one to make controllable calculations at any energy scale. However, the spectrum of particles in this theory includes nonphysical massive ghosts. The instabilities of classical solutions, generated by these ghosts, look non-avoidable. However, our experience in cosmology shows that these instabilities show up only in the regions where the typical energies are of the Planck order of magnitude. The role of the ghosts and the emergence of instabilities in the superrenormalizable QG models were not explored at the same level of quality, but it is expected that the output may be, qualitatively, the same.

整体来看，当前微扰量子引力的研究现状和整个量子引力领域的情况十分相似: 我们有很多方案可以将引力的量子效应纳入已经发展成熟的量子场论微扰形式体系，但另一方面，至今仍不存在一个完美的量子引力模型。基于广义相对论的理论是描述低能引力量子效应的优秀候选 (参见有效量子引力相关章节)，但它在紫外区存在严重的不可重整化问题。另一方面，四阶导数引力模型是可重整的，还能在任意能标下开展可控计算。但该理论的粒子谱包含非物理的大质量鬼，这些鬼引发的经典解不稳定性看来无法避免。不过我们的宇宙学经验表明，这类不稳定性仅出现在典型能量达到普朗克量级的区域。超可重整量子引力模型中鬼的作用与不稳定性的产生目前还没有得到充分研究，但一般预期定性结论会和上述情况一致。

We can draw two main conclusions about the main problem of perturbative QG. First, this problem is the conflict between renormalizability and the lack of physical unitarity (which does not reduce to the unitarity of the S -matrix [108]) or instabilities of classical solutions owing to the presence of ghosts. Second, the resolution of this conflict does not look impossible, but it will (most likely, at least) require new ideas and new approaches. According to DeWitt and Molina-Paris [109], long ago W. Pauli remarked ”It will take somebody really smart” to construct a quantum theory of gravity. After many decades that passed after this prediction, we are in a position to say that it will take somebody really smart to find modifications of gravity explaining how nature prevents the ghosts to emerge. The opinion of the present author is that the problem of ghosts shows up in the perturbative approach, but its solution will, most likely, be found only beyond this framework.

关于微扰量子引力的核心问题，我们可以得出两个主要结论。第一，该问题是可重整性与物理幺正性缺失 (其不能简化为 S 矩阵的幺正性 [108]) 之间的冲突，或是鬼存在导致经典解不稳定的问题。第二，这一冲突并非无法解决，但 (至少极有可能) 需要新的思路与新的研究方法。根据德维特与莫利纳-帕里斯 [109] 的记载，早在多年前 W. 泡利就指出，构建引力的量子理论“得靠一位真正聪慧的人”。在这一预言过去数十年后，我们如今可以说: 要找到引力的修正方案，解释自然如何避免鬼出现，依然得靠一位真正聪慧的人。作者本人的观点是: 鬼问题虽在微扰框架中显现，但其解决方案，极有可能只能在该框架之外找到。

From the historical perspective, one of the main impacts of QG to quantum field theory was the development of the classical and quantum theory of gauge fields, which helped to develop the Yang-Mills theory and gave a theoretical basis to the standard model of particle physics. In this respect, it is sufficient to mention the work of De Witt [13]. It is quite possible that the further progress in QG will be based on the flux of ideas in the opposite direction, which is using methods that already exist in field theory, but are not sufficiently familiar to the QG community.

从历史角度来看，量子引力对量子场论的一项主要影响是推动了规范场经典与量子理论的发展，这助力了杨-米尔斯理论的建立，为粒子物理标准模型奠定了理论基础，仅需提及 De Witt 的开创性工作 [13] 就足以说明这一点。未来量子引力的进一步发展非常可能依赖于反向的思路流动，也就是应用场论中已经存在、但量子引力研究界还不够熟悉的方法。

As one can learn from the variety of approaches presented in this handbook, we have many theories of QG. The main problem is perhaps not the shortage of the theories, but that none of these theories can be currently tested in experiments and/or observations. However, the increasing volume and improving quality of the observational data in astrophysics and cosmology may help to close this gap, someday. Thus, at least part of the nowadays theoretical developments in QG may find their use in the future.

正如本手册中介绍的诸多方案所展现的，我们已经拥有了大量量子引力理论。核心问题或许不是理论数量不足，而是目前这些理论都无法通过实验或观测检验。不过，天体物理与宇宙学观测数据的体量不断增长、质量不断提升，未来某一天也许能填补这一空白。因此，如今量子引力领域至少有一部分理论研究成果，将来总会找到用武之地。

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